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Asymptotics of the Instantons of Painlevé I

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The 0-instanton solution of Painlevé I is a sequence $(u_{n,0})$ of complex numbers which appears universally in many enumerative problems in algebraic geometry, graph theory, matrix models, and 2-dimensional quantum gravity. The asymptotics of the 0-instanton $(u_{n,0})$ for large n were obtained by the third author using the Riemann–Hilbert approach. For $k=0,1,2,\ldots,$ the k-instanton solution of Painlevé I is a doubly indexed sequence $(u_{n,k})$ of complex numbers that satisfies an explicit quadratic nonlinear recursion relation. The goal of the paper is three-fold: (a) to compute the asymptotics of the 1-instanton sequence $(u_{n,1})$ to all orders in $1/n$ by using the Riemann–Hilbert method, (b) to present formulas for the asymptotics of $(u_{n,k})$ for fixed k and to all orders in $1/n$ using resurgent analysis, and (c) to confirm numerically the predictions of resurgent analysis. We point out that the instanton solutions display a new type of Stokes behavior, induced from the tritronquée Painlevé transcendents, and which we call the induced Stokes phenomenon. The asymptotics of the 2-instanton and beyond exhibits new phenomena not seen in 0 and 1-instantons, and their enumerative context is at present unknown.

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1 Introduction

1.1 The Painlevé I equation and its 0-instanton solution

The *Painlevé I equation*

$$-\tfrac{1}{6}u'' + u^2 = z \tag{1.1}$$

is a nonlinear differential equation with strong integrability properties that appears universally in various scaling problems; see, for example [15]. The formal power-series solution

$$u_0(z) = z^{1/2} \sum_{n=0}^{\infty} u_{n,0} z^{-5n/2} \in z^{1/2}\mathbb{C}[\![z^{-5/2}]\!] \tag{1.2}$$

and correspondingly the sequence $(u_{n,0})$ is the so-called 0-*instanton* solution of Painlevé I. Substituting (1.2) into (1.1) collecting terms, and normalizing by setting $u_{0,0} = 1$, implies that $(u_{n,0})$ satisfies the following quadratic recursion relation:

$$u_{n,0} = \frac{25(n-1)^2 - 1}{48} u_{n-1,0} - \frac{1}{2} \sum_{\ell=1}^{n-1} u_{\ell,0} u_{n-\ell,0}, \qquad u_{0,0} = 1. \tag{1.3}$$

The 0-instanton solution $(u_{n,0})$ of Painlevé I is a sequence which plays a crucial role in many enumerative problems in algebraic geometry, graph theory, matrix models, and 2-dimensional quantum gravity; see, for example [11, 16, 17]. The leading asymptotics of the 0-instanton sequence $(u_{n,0})$ for large n was obtained by the third author using the *Riemann–Hilbert approach*; see [22]. In [20] (see also [16, Appendix A]), asymptotics to all orders in n were obtained as follows:

$$u_{n,0} \sim A^{-2n+1/2} \Gamma\left(2n - \frac{1}{2}\right) \frac{S_1}{\pi \mathrm{i}} \left\{ 1 + \sum_{l=1}^{\infty} \frac{u_{l,1} A^l}{\prod_{k=1}^{l} (2n - 1/2 - k)} \right\}, \qquad n \to \infty. \tag{1.4}$$

In this expression, $u_{l,1}$ are the coefficients of the 1-instanton series (defined below), A is the *instanton action*

$$A = \tfrac{8\sqrt{3}}{5} \tag{1.5}$$

and

$$S_1 = -\mathrm{i}\frac{3^{1/4}}{2\sqrt{\pi}} \tag{1.6}$$

is a Stokes constant.

1.2 The Painlevé I equation and its k-instanton solution

In this paper, we study the asymptotics of the k-instanton solution of $(u_{n,k})$ the Painlevé I equation. The doubly indexed sequence $(u_{n,k})$ for $n, k = 0, 1, 2, \ldots$ can be defined concretely by the following quadratic recursion relation: for $k = 1$, we have

$$u_{n,1} = \frac{8}{25An}\left\{12\sum_{l=0}^{n-2} u_{l,1}u_{(n+1-l)/2,0} - \frac{25}{64}(2n-1)^2 u_{n-1,1}\right\}, \quad u_{0,1} = 1, \tag{1.7}$$

while for general $k \geq 2$ we have

$$u_{n,k} = \frac{1}{12(k^2-1)}\left\{12\sum_{l=0}^{n-3} u_{l,k}u_{(n-l)/2,0} + 6\sum_{m=1}^{k-1}\sum_{l=0}^{n} u_{l,m}u_{n-l,k-m} \right.$$
$$\left. - \frac{25}{64}(2n+k-4)^2 u_{n-2,k} - \frac{25}{16}Ak(k+2n-3)u_{n-1,k}\right\}. \tag{1.8}$$

It is understood here that $u_{n/2,0} = 0$, if n is not an even integer.

The above recursion defines $u_{n,k}$ in terms of previous $u_{n',k'}$ for $n' < n$ or $n' = n$ and $k' < k$. The paper is concerned with the asymptotics of the sequence $(u_{n,k})$ for fixed k and large n. As we shall see, resurgence analysis predicts that the asymptotics of $(u_{n,k})$ to all orders in $1/n$ is given in terms of three known sequences $(u_{l,k\pm 1})$, $(\mu_{l,k\pm 1})$, and $(\phi_{l,k-1})$ and two Stokes constants S_1 and S_{-1}; see Equation (1.15). The first constant S_1 is known from the asymptotics of the 0-instanton $(u_{n,0})$ and the second one appears in the asymptotics of $(u_{n,k})$ for $k \geq 2$ and its exact value is unknown at present. Two new features appear in the asymptotics of $(u_{n,k})$ for $k \geq 2$: the constant S_{-1} and the presence of $\log n$ terms. These features are absent in the asymptotics of $(u_{n,0})$ and $(u_{n,1})$.

Equation (1.8) defines but does not motivate the k-instanton solution $(u_{n,k})$ to the Painlevé I equation. The motivation comes from the so-called *trans-series solution* of the Painlevé I equation. Trans-series were introduced and studied by Écalle in the 1980s; [12–14]. The trans-series solution $u(z, C)$ of the Painlevé I equation is a formal power series in two variables z and C that is defined as follows. Substitute the following expression

$$u(z, C) = \sum_{k\geq 0} C^k u_k(z) \tag{1.9}$$

into (1.1) and collect the coefficients of C^k. It follows that $u_k(z)$ satisfy the following hierarchy of differential equations, which are nonlinear for $k=0$, linear homogeneous for $k=1$, and linear inhomogeneous for $k\geq 2$:

$$\begin{aligned} -\tfrac{1}{6}u_0''+u_0^2&=z \\ -\frac{1}{6}u_k''+\sum_{k'=0}^{k}u_{k'}u_{k-k'}&=0, \quad k\geq 1. \end{aligned} \tag{1.10}$$

$u_k(z)$ is known as the k-instanton solution of (1.1), and it has the following structure:

$$u_k(z)=z^{1/2}\mathrm{e}^{-kAz^{5/4}}\phi_k(z), \tag{1.11}$$

where A is given in (1.5) and

$$\phi_k(z)=z^{-5k/8}\sum_{n\geq 0}u_{n,k}z^{-5n/4}. \tag{1.12}$$

Since C is arbitrary we can normalize

$$u_{0,1}=1. \tag{1.13}$$

This motivates our definition of $(u_{n,k})$. The trans-series (1.9) is what is called a *proper* trans-series, in the sense that all the exponentials appearing in it are small when $z\to\infty$ along the direction $\mathrm{Arg}(z)=0$. Therefore, the instanton solutions $u_k(z)$ give exponentially small corrections to the asymptotic expansion (1.2) and they can be used to construct actual solutions of the Painlevé I equation in certain sectors; see [4].

The instanton solutions of Painlevé I also have an important physical interpretation in the context of two-dimensional quantum gravity and noncritical string theory. As is well known (see, for example [11] and references therein), the Painlevé I equation appears in the so-called double-scaling limit of random matrices with a polynomial potential, and it is interpreted as the equation governing the specific heat of a noncritical string theory. The asymptotic expansion of the 0-instanton solution (1.2) is nothing but the genus or perturbative expansion of this string theory, and $z^{-5/4}$ is interpreted as the string coupling constant. The instanton solutions to Painlevé I correspond to non-perturbative corrections to this expansion. In the context of matrix models, they can be interpreted as the double-scaling limit of matrix model instantons, which are obtained by eigenvalue tunneling [10, 23]. In the context of noncritical string theory, they are due to a special type of D-branes called ZZ branes [1, 24]. The asymptotic behavior of the k-instanton solution is then important in order to understand the full nonperturbative structure of these theories.

1.3 The predictions of resurgence for the k-instanton asymptotics

Our paper consists of three parts.

(a) A proof of the all-orders asymptotics of the 0 and 1-instantons of Painlevé I, using the Riemann–Hilbert approach.

(b) A resurgence analysis of the k-instantons of Painlevé I and their asymptotics to all orders.

(c) A numerical confirmation of the predictions of the resurgence analysis.

In this section, we state the predictions of resurgence analysis for the asymptotics of $(u_{n,k})$ for fixed k and large n. Recall from Equation (1.4) that the $1/n^k$ asymptotics of $(u_{n,0})$ involve the 1-instanton $u_{\ell,1}$ for $\ell \leq k$, and a Stokes constant S_1. Likewise, the asymptotics of $(u_{n,k})$ for fixed $k \geq 1$ and large n involves two auxilliary doubly indexed sequences $(\mu_{n,k})$ and $(\nu_{n,k})$ and two Stokes constants S_1 and S_{-1}. The sequence $(\nu_{n,k})$ is proportional to the instanton sequence

$$\nu_{n,k} = \begin{cases} \dfrac{16}{5A} k u_{n,k}, & k > 0, \\ 0, & k = 0. \end{cases} \tag{1.14}$$

The sequence $(\mu_{n,k})$ is defined by

$$\begin{aligned}
\mu_{n,1} &= (-1)^n u_{n,1}, \\
\mu_{2n,2} &= -\sum_{l=0}^{n-2} \mu_{2l,2} u_{n-l,0} - \sum_{l=0}^{2n} (-1)^l u_{l,1} u_{2n-l,1} + \frac{25}{192} (2n-1)^2 \mu_{2(n-1),2}, \\
\mu_{2n+1,2} &= 0, \\
\mu_{n,3} &= \frac{8}{25A(n+1)} \Bigg\{ -\frac{25}{64} (2n+1)^2 \mu_{n-1,3} + 12 \sum_{l=0}^{n-2} u_{(n+1-l)/2,0} \mu_{l,3} \\
&\qquad + 12 \sum_{m=1}^{2} \sum_{l=0}^{n+1} u_{n+1-l,m} \mu_{l,3-m} + \frac{25}{16} (2n+1) \nu_{n,1} + \frac{25A}{8} \nu_{n+1,1} \Bigg\}
\end{aligned}$$

for $k=1,2,3$, and by

$$\mu_{n,k}=\frac{1}{12(k-1)(k-3)}\Bigg\{12\sum_{l=0}^{n-3}\mu_{l,k}u_{(n-l)/2,0}+12\sum_{m=1}^{k-1}\sum_{l=0}^{n}\mu_{l,m}u_{n-l,k-m}-\frac{25}{64}(2n+k-4)^2\mu_{n-2,k}$$
$$-\frac{25}{16}Ak(k+2n-3)\mu_{n-1,k}+\frac{25}{16}(k+2n-4)\nu_{n-1,k-2}+\frac{25A}{8}(k-2)\nu_{n,k-2}\Bigg\}$$

for $k\geq 4$. We apologize for the lengthy formulas that define the doubly indexed sequences $(\mu_{n,k})$ and $(\nu_{n,k})$. As in Section 1.2, there is a simple resurgence analysis explanation of these sequences given in detail in Section 4. An analysis based on trans-series solutions and resurgent properties suggests the following asymptotic result for the coefficients $u_{n,k}$, for fixed $k\geq 1$, in the limit $n\to\infty$:

$$u_{n,k}\sim_n A^{-n+1/2}\frac{S_1}{2\pi\mathrm{i}}\Gamma\left(n-\frac{1}{2}\right)$$
$$\times\left\{(k+1)u_{0,k+1}+(-1)^n\mu_{0,k+1}+\sum_{l=1}^{\infty}\frac{((k+1)u_{l,k+1}+(-1)^{n+l}\mu_{l,k+1})A^l}{\prod_{m=1}^{l}(n-1/2-m)}\right\}$$
$$+(-1)^n(k-1)A^{-n-2/3}\frac{S_{-1}}{2\pi\mathrm{i}}\Gamma\left(n+\frac{1}{2}\right)\left\{u_{0,k-1}+\sum_{l=1}^{\infty}\frac{u_{l,k-1}(-A)^l}{\prod_{m=1}^{l}(n+1/2-m)}\right\}$$
$$-(-1)^nA^{-n-1/2}\frac{S_1}{2\pi\mathrm{i}}\Gamma\left(n+\frac{1}{2}\right)(\log n-\log A)\left\{\nu_{0,k-1}+\sum_{l=1}^{\infty}\frac{\nu_{l,k-1}(-A)^l}{\prod_{m=1}^{l}(n+1/2-m)}\right\}$$
$$-(-1)^nA^{-n-1/2}\frac{S_1}{2\pi\mathrm{i}}\Gamma\left(n+\frac{1}{2}\right)$$
$$\times\left\{\left(\psi\left(n+\frac{1}{2}\right)-\log n\right)\nu_{0,k-1}+\sum_{l=1}^{\infty}\frac{\psi(n+1/2-l)-\log n}{\prod_{m=1}^{l}(n+1/2-m)}\nu_{l,k-1}(-A)^l\right\},\tag{1.15}$$

where

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$$

is the logarithmic derivative of the Γ function; see [25]. In the above equation, the convention is that $u_{n,k}=\nu_{n,k}=0$ for $k=0$.

The above formula gives an asymptotic expansion for $u_{n,k}$ for n large, involving the coefficients $u_{l,k\pm1}$, $\mu_{l,k\pm1}$, and $\nu_{l,k\pm1}$. Up to the overall factor $A^{-n}\Gamma(n+1/2)$, this expansion involves terms of the form $1/n^l$ and $\log n/n^l$. In order to get the mth first terms of this asymptotic expansion for $u_{n,k}$, we only need to know the coefficients $u_{l,k\pm1}$, $\mu_{l,k\pm1}$, and $\nu_{l,k\pm1}$ with l up to m. In particular, (1.15) gives an efficient method to obtain the asymptotic behavior of the $u_{n,k}$ for fixed k at large n.

As a concrete and important example which also clarifies the above remarks, let us look at $k=1$. In this case, only the first line of Equation (1.15) contributes and resurgence analysis predicts that for even n, we have

$$u_{2n,1} \sim A^{-2n+1/2}\frac{S_1}{2\pi \mathrm{i}}\Gamma\left(2n-\frac{1}{2}\right)\left\{2u_{0,2}+\mu_{0,2}+\sum_{l=1}^{\infty}\frac{(2u_{l,2}+(-1)^l\mu_{l,2})A^l}{\prod_{m=1}^{l}(2n-1/2-m)}\right\},\quad n\to\infty, \tag{1.16}$$

while for odd n we have

$$u_{2n+1,1} \sim A^{-2n-1/2}\frac{S_1}{2\pi \mathrm{i}}\Gamma\left(2n+\frac{1}{2}\right) \times\left\{2u_{0,2}-\mu_{0,2}+\sum_{l=1}^{\infty}(2u_{l,2}-(-1)^l\mu_{l,2})\lambda^l\prod_{m=1}^{l}(2n+1/2-m)\right\},\quad n\to\infty. \tag{1.17}$$

This prediction will be proved in Section 3 by using the Riemann–Hilbert approach. The coefficients $u_{0,2}$ and $\mu_{0,2}$ can easily be calculated from the recursion (1.8):

$$u_{0,2}=\tfrac{1}{6},\quad \mu_{0,2}=-1. \tag{1.18}$$

In fact one can obtain closed formulae for $u_{0,k}$ and $\mu_{0,k}$ for all k (see (4.34) and (4.36), respectively). The explicit result (1.18) gives the leading asymptotics,

$$u_{n,1}\sim\left(\frac{1}{3}-(-1)^n\right)A^{-n+1/2}\frac{S_1}{2\pi \mathrm{i}}\Gamma\left(n-\frac{1}{2}\right),\quad n\to\infty. \tag{1.19}$$

Concrete examples of the implications of (1.15) for the asymptotics of the $u_{n,k}$ are given in Section 5, where the formula is tested numerically.

For some partial results on the Painlevé I equation and the asymptotics of the coefficients of the 0-instanton solution, see also [4, 5, 9]. To the best of our knowledge, a rigorous computation of the Stokes constant S_1 has only been achieved via the Riemann–Hilbert approach [22] or its earlier version—the isomonodromy method [21, 27]. If at all possible, a computation of the constants S_1 and S_{-1} using resurgence analysis would be very interesting.

1.4 The asymptotics of the 1-instanton solution via the Riemann–Hilbert method

Recall from Equations (1.10) and (1.11) that the generating series $u_1(z)$ of the 1-instanton $(u_{n,1})$ solution to Painlevé I satisfies the differential equation

$$u_1''(z) = 12u_1(z)u_0(z), \tag{1.20}$$

where $u_0(z)$ is the generating series of the 0-instanton solution $(u_{n,0})$ solution to Painlevé I. There are five explicit *tritronquées* meromorphic solutions of Painlevé I equation asymptotic to $u_0(z)$ in appropriate sectors in the complex plane, discussed in Section 2.1. To study the asymptotics of the 1-instanton $(u_{n,1})$, consider the linear homogenous differential equation

$$v'' = \gamma uv, \tag{1.21}$$

where γ is a constant and u is a tritronquée solution to Painlevé I. It is easy to see that Equation (1.21) has two linearly independent formal power series solutions $v_f^{\pm}$ of the form

$$v_f^{\pm}(z) = z^{-1/8}\mathrm{e}^{\pm 4(\sqrt{\gamma}/5)z^{5/4}} \sum_{n=0}^{\infty} b_n^{\pm}(\gamma) z^{-5n/4}, \tag{1.22}$$

where $b_n^{\pm} = b_n^{\pm}(\gamma)$ is given by

$$b_n^{\pm} = \frac{1}{2n}\left\{ \frac{5b_{n-1}^{\pm}}{4\sqrt{\gamma}}\left(n-\frac{1}{10}\right)\left(n-\frac{9}{10}\right) \mp \frac{4\sqrt{\gamma}}{5}\sum_{m=1}^{[(n+1)/2]} u_{m,0} b_{n+1-2m}^{\pm} \right\}, \quad b_0^{\pm} = 1. \tag{1.23}$$

Note that $b_n^{\pm}(\gamma)$ is a polynomial in $\gamma^{1/2}$ and $b_n^{-}(12) = u_{n,1}$. The next theorem gives the asymptotic expansion of b_n^{-} when $\gamma > 3$. Let

$$B = \tfrac{4}{5}\sqrt{\gamma}. \tag{1.24}$$

Theorem 1.1. For n large and $\gamma > 3$, we have

$$b_n^{-}(\gamma) = \frac{4\cdot 3^{1/4}}{25\pi^{3/2}} \frac{A(1+(-1)^n) - 2B(1-(-1)^n)}{4B^2 - A^2} \gamma A^{-n-1/2} \Gamma\left(n-\frac{1}{2}\right)(1+\mathcal{O}(n^{-1})). \tag{1.25}$$

□

The proof of the theorem uses explicitly the five *tritronquées* solutions of Painlevé I equation and the asymptotic expansion of their difference in five sectors of the complex plane, computed by the Riemann–Hilbert approach to Painlevé I. An analytic novelty of Theorem 1.1 is the rigorous computation of a Stokes constant which is independent of γ, when $\gamma > 3$.

Remark 1.1. There are two special values of γ in Equation (1.21). When $\gamma = 12$ and v is a solution of Equation (1.21), it follows that v is the 1-instanton solution of Painleve I and Theorem 1.1 implies Equations (1.16) and (1.17). When $\gamma = 3/4$ and v is a solution of (1.21), then $\tilde{v} = -2v'/v$ is a solution of the Riccati equation

$$2\tilde{v}' - \tilde{v}^2 + 3u = 0 \tag{1.26}$$

studied in [17]. □

2 Asymptotics of the 0-instanton of Painlevé I

In this section, we review and extend the calculation of the asymptotics of the 0-instanton sequence $(u_{n,0})$ presented in [22]:

(a) Use the five tritronquée solutions to Painlevé I and their analytic properties obtained by the Riemann–Hilbert method, as an input.

(b) No two tritronquée solutions are equal on a sector, however their difference is exponentially small. Using the five tritronquée solutions and their differences, define a piece-wise analytic function in the complex plane which has (1.2) as its uniform asymptotics in the neighborhood of infinity.

(c) Apply a mock version of the Cauchy integral formula to obtain an exact integral formula for $u_{n,0}$ in terms of the jumps of the piece-wise analytic function defined above. The knowledge of the explicit value of the relevant Stokes' multiplier (available due to the Riemann–Hilbert analysis) yields the large n asymptotics of $u_{n,0}$. In fact, we extend the result of [22] and we obtain the asymptotics to all orders in $1/n$.

Note that this method consists of working entirely in the z-plane (and not in the Borel plane), in all sectors simultaneously, and our glued function is only piece-wise analytic. This method is different from the method of Borel transforms analyzed in detail in [5, 9].

2.1 A review of the tritronquée solutions of Painlevé I

In this section, to simplify our notation, we will replace $u_{n,0}$ by a_n, and use the symbol u_n *not* for the nth term in the formal trans-series expansion, but for certain exact solutions of the first Painlevé equation specified below.

Recall that equation PI (1.1) admits a formal 0-parameter solution with the power expansion $u_f(z)$ (cf. (1.2)),

$$u_f(z) = z^{1/2}\sum_{n=0}^{\infty} a_n z^{-5n/2}, \quad a_0 = 1, \quad a_1 = -\frac{1}{48}, \quad a_{n+1} = \frac{25n^2-1}{48}a_n - \frac{1}{2}\sum_{m=1}^{n} a_m a_{n+1-m}. \tag{2.1}$$

Remark 2.1. In [22], the Painlevé I was studied in the following form:

$$y_{xx} = 6y^2 + x. \tag{2.2}$$

The change of variables $u = 6^{2/5}y$ and $z = \mathrm{e}^{-\mathrm{i}\pi}6^{-1/5}x$ transforms (1.1) to (2.2). Comparing with [22], we also find more convenient to modify the indices for the tritronquée solutions. □

Using the Riemann–Hilbert approach (see [15, Chapter 5.2] and also [22]), it can be shown that there exist five different genuine meromorphic solutions of (1.1) with the asymptotic power expansion (2.1) in one of the sectors of the z-complex plane of opening $8\pi/5$, see Figure 1:

$$\begin{aligned} &u_0(z) \sim u_f(z), \quad z \longrightarrow \infty, \quad \arg z \in \left(-\frac{6\pi}{5}, \frac{2\pi}{5}\right), \\ &u_k(z) = \mathrm{e}^{-\mathrm{i}(8\pi/5)k}u_0(\mathrm{e}^{-\mathrm{i}(4\pi/5)k}z) \sim u_f(z), \quad z \longrightarrow \infty, \quad \arg z \in \left(-\frac{6\pi}{5} + \frac{4\pi}{5}k, \frac{2\pi}{5} + \frac{4\pi}{5}k\right), \end{aligned} \tag{2.3}$$

$$u_{k+5}(z) = u_k(z), \tag{2.4}$$

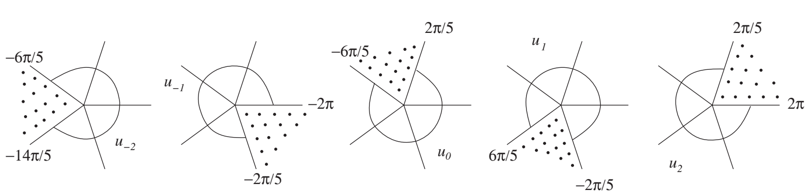

Fig. 1. The sectors of the z-complex plane where the tritronquée solutions (2.3) u_{-2}, u_{-1}, u_0, u_1, and u_2 are represented by the formal series u_f.In the dotted sectors, the asymptotics at infinity of the tritronquée solutions is elliptic.

where in $u_f(z)$ appearing in (2.3) the branches of $z^{1/2}$ and $z^{-5n/2}$ are defined according to the rule,

$$z^{1/2} = \sqrt{|z|}\mathrm{e}^{\arg z/2}, \quad \arg z \in \left(-\frac{6\pi}{5} + \frac{4\pi}{5}k, \frac{2\pi}{5} + \frac{4\pi}{5}k\right), \quad z^{-5n/2} = (z^{1/2})^{-5n}. \tag{2.5}$$

The asymptotic formula (2.3) is understood in the usual sense. That is, for every $\epsilon > 0$ and natural number N there exist positive constants $C_{N,\epsilon}$ and $R_{N,\epsilon}$ (depending on N and ϵ only) such that,

$$\begin{gathered} \left| u_k(z) - z^{1/2}\sum_{n=0}^{N} a_n z^{-5n/2}\right| < C_{N,\epsilon}|z|^{-(5N+4)/2}, \\ \forall z : |z| \geq R_{N,\epsilon}, \quad \arg z \in \left[-\frac{6\pi}{5} + \frac{4\pi}{5}k + \epsilon, \frac{2\pi}{5} + \frac{4\pi}{5}k - \epsilon\right]. \end{gathered} \tag{2.6}$$

Here, the branches of $z^{1/2}$ and $z^{-5n/2}$ are again defined according to (2.5). Estimate (2.6) is proved in [22].

Remark 2.2. We note that estimate (2.6) implies, in particular, that each tritronquée solution $u_k(z)$ might have in the sector $[-\frac{6\pi}{5} + \frac{4\pi}{5}k + \epsilon, \frac{2\pi}{5} + \frac{4\pi}{5}k - \epsilon]$ only finitely many poles which lie inside the circle of radius $R_{0,\epsilon}$. We also note that the number $R_{N,\epsilon}$ in estimate (2.6) can be replaced by $R_{0,\epsilon}$ for every N. □

Furthermore, in [22] it is shown that the exponential small difference between the tritronquée solutions $u_k(z)$ and $u_{k+1}(z)$, within the common sector where they have the identical asymptotics $u_f(z)$ in all orders, admits the following explicit asymptotic description:

$$\begin{aligned} &z \longrightarrow \infty, \ \arg z \in \left(-\frac{2\pi}{5}, \frac{2\pi}{5}\right) : u_1(z) - u_0(z) = \mathrm{i}\frac{3^{1/4}}{2\sqrt{\pi}} z^{-1/8}\mathrm{e}^{-(8\sqrt{3}/5)z^{5/4}}(1 + \mathcal{O}(z^{-5/4})), \\ &z \longrightarrow \infty, \ \arg z \in \left(-\frac{2\pi}{5} + \frac{4\pi}{5}k, \frac{2\pi}{5} + \frac{4\pi}{5}k\right) : u_{k+1}(z) - u_k(z) \\ &\quad = \mathrm{e}^{\mathrm{i}(\pi/2)(k+1)}\frac{3^{1/4}}{2\sqrt{\pi}} z^{-1/8}\mathrm{e}^{(-1)^{k+1}(8\sqrt{3}/5)z^{5/4}}(1 + \mathcal{O}(z^{-5/4})). \end{aligned} \tag{2.7}$$

Here again the asymptotic relations mean that the differences $u_{k+1}(z) - u_k(z)$ admit the representation,

$$u_{k+1}(z) - u_k(z) = \mathrm{e}^{\mathrm{i}(\pi/2)(k+1)}\frac{3^{1/4}}{2\sqrt{\pi}} z^{-1/8}\mathrm{e}^{(-1)^{k+1}(8\sqrt{3}/5)z^{5/4}}(1 + r(z)), \tag{2.8}$$

with the error term $r(z)$ satisfying the estimate (cf. (2.6)),

$$|r(z)| < C_\epsilon |z|^{-5/4},$$
$$\forall z: |z| \geq R_\epsilon, \quad \arg z \in \left[-\frac{2\pi}{5} + \frac{4\pi}{5}k + \epsilon, \frac{2\pi}{5} + \frac{4\pi}{5}k - \epsilon\right], \tag{2.9}$$

with the positive constants C_ϵ and R_ϵ depending on ϵ only. In fact, we can take $R_\epsilon = R_{0,\epsilon}$.

When dealing with the tritronquée solutions $u_k(x)$, it is convenient to assume that k can take any integer value, simultaneously remembering that $u_{k+5}(x) = u_k(x)$, see (2.3). In other words, in the notation $u_k(x)$, we shall assume that

$$k \in \mathbb{Z}, \bmod 5, \tag{2.10}$$

unless k is particularly specified, as in (2.16).

Remark 2.3. The existence of the tritronquée solutions $u_k(x)$ can be proved without use of the Riemann–Hilbert method (see, e.g. [20] and references therein). Indeed, these solutions had already been known to Boutroux. The Riemann–Hilbert method is needed for the exact evaluation of the pre-exponent numerical coefficient in the jump-relations (2.7), that is, for an explicit description of the *quasi-linear Stokes' phenomenon* exhibited by the first Painlevé equation. □

2.2 Gluing the five tritronquée solutions together

The main technical part of the approach of [22] to the 0-instanton asymptotics is a piecewise meromorphic function with uniform asymptotics (2.1) at infinity. This function is constructed from the collection of the tritronqueée solutions $u_k(z)$, $k=0, \pm 1, \pm 2$, as follows.

First, observe that the change of independent variable $z = t^2$ turns (2.1) into the nonbranching series,

$$\hat{u}_f(t) := u_f(t^2) = \sum_{n=0}^{\infty} a_n t^{-5n+1}. \tag{2.11}$$

Multiplying (2.11) by t^{5N-2}, we obtain the formal series

$$\hat{u}_f^{(N)}(t) \equiv \hat{u}_f(t) t^{5N-2} = P_{5N-1}(t) + a_N t^{-1} + \sum_{n=N+1}^{\infty} a_n t^{-5(n-N)-1}, \quad P_{5N-1}(t) = \sum_{n=1}^{N} a_{N-n} t^{5n-1}. \tag{2.12}$$

Let us perform the similar operation with the solutions $u_k(z)$

$$\hat{u}_k(t) = u_k(t^2), \tag{2.13}$$

$$\hat{u}_k^{(N)}(t) = \hat{u}_k(t)t^{5N-2} - P_{5N-1}(t). \tag{2.14}$$

Observe that,

$$\hat{u}_k^{(N)}(t) = a_N t^{-1} + \mathcal{O}_N(t^{-6}), \quad t \longrightarrow \infty, \quad \arg t \in \left(-\frac{3\pi}{5} + \frac{2\pi}{5}k, \frac{\pi}{5} + \frac{2\pi}{5}k\right).$$

The symbol $\mathcal{O}_N$ indicates that the relevant positive constants in the estimate depend on N. More precisely, estimate (2.6) implies that

$$\begin{gathered} |\hat{u}_k^{(N)}(t) - a_N t^{-1}| < C_{N,\epsilon}|t|^{-6}, \\ \forall t : |t| \geq R_{0,\epsilon}^{1/2}, \quad \arg t \in \left[-\frac{3\pi}{5} + \frac{2\pi}{5}k + \epsilon, \frac{\pi}{5} + \frac{2\pi}{5}k - \epsilon\right]. \end{gathered} \tag{2.15}$$

Moreover, in view of Remark 2.2, we conclude that each function $\hat{u}_k^{(N)}(t)$ might have in the sector $[-\frac{3\pi}{5} + \frac{2\pi}{5}k + \epsilon, \frac{\pi}{5} + \frac{2\pi}{5}k - \epsilon]$ only finitely many poles whose number does not depend on N, and they all lie inside the circle of radius $R_{0,\epsilon}^{1/2}$. Indeed, by (2.14), the possible poles must coincide, for every N, with the possible poles of the corresponding function $\hat{u}_k(t)$.

Put

$$\hat{u}^{(N)}(t) = \hat{u}_k^{(N)}(t), \quad \arg t \in \left(-\frac{2\pi}{5} + \frac{2\pi}{5}k, \frac{2\pi}{5}k\right), \quad k = 0, \pm 1, \pm 2. \tag{2.16}$$

These equations determine $\hat{u}^{(N)}(t)$ as a sectorially meromorphic function $\hat{u}^{(N)}(t)$ discontinuous across the rays $\arg t = \frac{2\pi}{5}k$, $k = 0, \pm 1, \pm 2$, (see Figure 2).

Let us choose and then *fix* the parameter $\epsilon \equiv \epsilon_0$ in such a way that the closure of each sector $(-\frac{2\pi}{5} + \frac{2\pi}{5}k, \frac{2\pi}{5}k)$ is included in the corresponding sector $[-\frac{3\pi}{5} + \frac{2\pi}{5}k + \epsilon, \frac{\pi}{5} + \frac{2\pi}{5}k - \epsilon]$. Then, the function $\hat{u}^{(N)}(t)$ would have no more than finitely many poles in the whole complex plane, the number of poles would be the same for all N, and they all lie inside the circle of radius

$$R_0 = R_{0,\epsilon_0}^{1/2}. \tag{2.17}$$

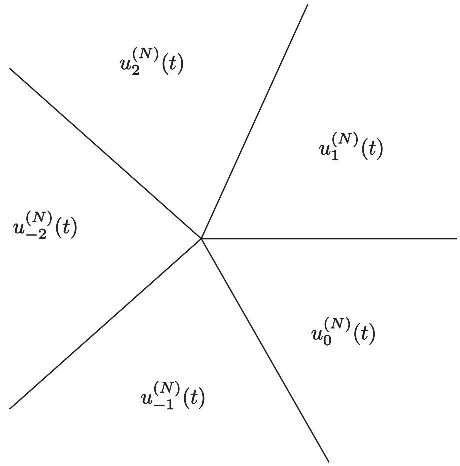

Fig. 2. The sectorially meromorphic function $\hat{u}^{(N)}(t)$ (2.16).

In addition, the following *uniform* asymptotics at infinity takes place:

$$\hat{u}^{(N)}(t) = a_N t^{-1} + \mathcal{O}_N(t^{-6}), \quad t \longrightarrow \infty. \tag{2.18}$$

The exact meaning of this estimate is the existence of the positive constant $C_N (\equiv C_{N,\epsilon_0})$ such that,

$$\begin{gathered} |\hat{u}^{(N)}(t) - a_N t^{-1}| < C_N |t|^{-6}, \\ \forall t : |t| > R_0. \end{gathered} \tag{2.19}$$

Here, R_0 is defined in (2.17), and it does not depend on N. We also remind the reader that the circle of radius R_0 contains all the possible poles of $\hat{u}^{(N)}(t)$.

The jumps of the function $\hat{u}^{(N)}(t)$ across the rays $\arg t = \frac{2\pi}{5} k$, $k = 0, \pm 1, \pm 2$, oriented towards infinity, are described by the equations,

$$\hat{u}_+^{(N)}(t) - \hat{u}_-^{(N)}(t) = \hat{U}^{(N)}(t), \quad \arg t = \frac{2\pi}{5} k, \quad k = 0, \pm 1, \pm 2, \tag{2.20}$$

where $\hat{u}_+^{(N)}(t)$ and $\hat{u}_-^{(N)}(t)$ are the limits of $\hat{u}^{(N)}(t)$ as we approach the rays from the left and from the right, respectively. (We note that $\hat{u}_6^{(N)}(t) = \hat{u}_1^{(N)}(t)$, as it follows from (2.3)). By virtue of (2.7), the jump functions $\hat{U}^{(N)}(t)$ satisfy the estimate,

$$\hat{U}^{(N)}(t)|_{\arg t = (2\pi/5)k} = \mathrm{e}^{\mathrm{i}(\pi/2)(k+1)} \frac{3^{1/4}}{2\sqrt{\pi}} t^{5N-9/4} \mathrm{e}^{(-1)^{k+1}(8\sqrt{3}/5)t^{5/2}} (1 + \mathcal{O}(t^{-5/2})), \tag{2.21}$$

which, again, means (cf. (2.8) and (2.9)) that

$$\hat{U}^{(N)}(t)|_{\arg t=(2\pi/5)k} = \mathrm{e}^{\mathrm{i}(\pi/2)(k+1)} \frac{3^{1/4}}{2\sqrt{\pi}} t^{5N-9/4} \mathrm{e}^{(-1)^{k+1}(8\sqrt{3}/5)t^{5/2}} (1 + r(t)), \tag{2.22}$$

with

$$|r(t)| < C|t|^{-5/2}, \quad \forall t : |t| > R_0, \quad \arg t = \frac{2\pi}{5}k,$$

where the constant $C(\equiv C_{\epsilon_0})$, similar to R_0, is a numerical constant not depending on N.

2.3 An integral formula for the 0-instanton coefficients and their asymptotic expansion

Consider the integral,

$$\frac{1}{2\pi \mathrm{i}} \oint_{|t|=R} \hat{u}^{(N)}(t)\, \mathrm{d}t,$$

of the function $\hat{u}^{(N)}$ along the circle of radius R centered at the origin and counter-clockwise oriented. For sufficiently large R, we can apply the integrand estimate (2.19). Noting that the constant C_N is the same along the whole circle, we conclude that

$$\frac{1}{2\pi \mathrm{i}} \oint_{|t|=R} \hat{u}^{(N)}(t)\, \mathrm{d}t = a_N + \mathcal{O}_N(R^{-5}), \tag{2.23}$$

that is,

$$\left| \frac{1}{2\pi \mathrm{i}} \oint_{|t|=R} \hat{u}^{(N)}(t)\, \mathrm{d}t - a_N \right| < C_N R^{-5}, \quad \forall R > R_0. \tag{2.24}$$

On the other hand, since $\hat{u}^{(N)}(t)$ can have only a finite number of poles lying inside the circle with the radius R_0, the circular contour of integration can be deformed to the sum of the circle of smaller radius $|t| = \rho \geq R_0$, still containing inside all the possible poles of $\hat{u}^{(N)}(t)$, and positive and negative sides of segments of the rays $\arg t = \frac{2\pi}{5}k$ (see Figure 3).

In other words, taking into account (2.20), we have that

$$\begin{aligned} a_N &= \frac{1}{2\pi \mathrm{i}} \oint_{|t|=R\gg 1} \hat{u}^{(N)}(t)\, \mathrm{d}t + \mathcal{O}_N(R^{-5}) \\ &= -\frac{1}{2\pi \mathrm{i}} \sum_{k=-2}^{2} \int_{\rho \mathrm{e}^{\mathrm{i}(2\pi/5)k}}^{R\mathrm{e}^{\mathrm{i}(2\pi/5)k}} \hat{U}^{(N)}(t)\, \mathrm{d}t + \frac{1}{2\pi \mathrm{i}} \oint_{|t|=\rho} \hat{u}^{(N)}(t)\, \mathrm{d}t + \mathcal{O}_N(R^{-5}) \\ &= -\frac{1}{2\pi \mathrm{i}} \sum_{k=-2}^{2} \int_{\rho \mathrm{e}^{\mathrm{i}(2\pi/5)k}}^{R\mathrm{e}^{\mathrm{i}(2\pi/5)k}} \hat{U}^{(N)}(t)\, \mathrm{d}t + \frac{1}{2\pi \mathrm{i}} \oint_{|t|=\rho} (\hat{u}^{(N)}(t) + P_{5N-1}(t))\, \mathrm{d}t + \mathcal{O}_N(R^{-5}), \end{aligned} \tag{2.25}$$

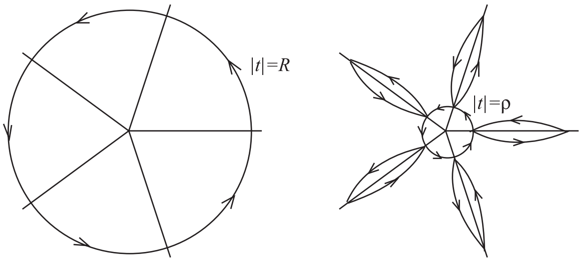

Fig. 3. Deformation of the contour of integration for computation of the 0-instanton N-large asymptotics.

where we note that the integral of the polynomial $P_{5N-1}(t)$ can be indeed added to the right-hand side since it is zero. Let us now note that from definition (2.16) of the function $\hat{u}^{(N)}(t)$ it follows that

$$\hat{u}^{(N)}(t) + P_{5N-1}(t) = \hat{u}(t)t^{5N-1}, \tag{2.26}$$

where $\hat{u}(t)$ is defined by the equations (cf. (2.16)),

$$\hat{u}(t) = \hat{u}_k(t), \quad \arg t \in \left(-\frac{2\pi}{5} + \frac{2\pi}{5}k, \frac{2\pi}{5}k\right), \quad k = 0, \pm 1, \pm 2. \tag{2.27}$$

By exactly the same reasons as in the case of estimate (2.19), we conclude from (2.6) that

$$|\hat{u}(t)| < C^{(1)}|t|, \quad \forall t : |t| > R_0, \tag{2.28}$$

with a numerical constant $C^{(1)}$ this time independent of N. (The constant $C^{(1)}$ can be taken equal to $C^{(1)} = 1 + \frac{C_{0,\epsilon_0}}{R_0^5}$.) Estimate (2.28), together with (2.26) implies the inequality,

$$|\hat{u}^{(N)}(t) + P_{5N-1}(t)|_{|t|=\rho} < C^{(1)}\rho^{5N-1},$$

where ρ is assumed to be a fixed positive number satisfying

$$R_0 \le \rho < R.$$

Therefore, formula (2.25) can be transformed into the formula,

$$a_N = -\frac{5}{2\pi \mathrm{i}} \int_{\rho}^{R} \hat{U}^{(N)}(t)\,\mathrm{d}t + r_N^{(1)}(\rho) + r_N^{(2)}(R), \tag{2.29}$$

where the error terms $r_N^{(1)}(\rho)$ and $r_N^{(2)}(R)$ satisfy the estimates,

$$|r_N^{(1)}(\rho)| < C^{(1)}\rho^{5N}, \quad \forall N \geq 1 \tag{2.30}$$

and

$$|r_N^{(2)}(R)| < C_N R^{-5}, \quad \forall R > R_0, \tag{2.31}$$

respectively. We remind the reader that the constants C_N depends on N only while the constant $C^{(1)}$ is a numerical constant independent of N. In derivation of (2.29), we also took into account symmetries (2.3) which allowed us to replace the sum of integrals from (2.25) by a single integral.

From (2.22) it follows that the integrand in (2.29) satisfies the asymptotic equation,

$$\hat{U}^{(N)}(t)|_{\arg t=0} = \mathrm{e}^{\mathrm{i}(\pi/2)(k+1)}\frac{3^{1/4}}{2\sqrt{\pi}}\, t^{5N-9/4}\mathrm{e}^{-(8\sqrt{3}/5)t^{5/2}}(1+r(t)), \tag{2.32}$$

with

$$|r(t)| < Ct^{-5/2}, \quad \forall t : t > R_0.$$

In particular, this means that the integral of $\hat{U}^{(N)}(t)$ along the half line $[\rho, \infty)$ converges. The coefficient a_N and the term $r^{(1)}(\rho)$ in (2.29) do not depend . Indeed, we have that

$$r^{(1)}(\rho) = \frac{1}{2\pi\mathrm{i}}\oint_{|t|=\rho}(\hat{u}^{(N)}(t) + P_{5N-1}(t))\,\mathrm{d}t$$

on R while the term $r^{(2)}$, in view of (2.31), vanishes as $R \to \infty$. Therefore, sending $R \to \infty$ in (2.29) (and keeping N fixed) we arrive at the equation

$$a_N = -\frac{5}{2\pi\mathrm{i}}\int_\rho^\infty \hat{U}^{(N)}(t)\,\mathrm{d}t + r_N^{(1)}(\rho), \tag{2.33}$$

which in turn can be transformed as follows:

$$\begin{aligned} a_N = &-\frac{5}{2\pi\mathrm{i}}\int_\rho^\infty \hat{U}^{(N)}(t)\,\mathrm{d}t + r_N^{(1)}(\rho) - \frac{5}{2\pi\mathrm{i}}\int_\rho^\infty \mathrm{e}^{\mathrm{i}\pi/2}\frac{3^{1/4}}{2\sqrt{\pi}}\, t^{5N-9/4}\mathrm{e}^{-(8\sqrt{3}/5)t^{5/2}}\,\mathrm{d}t \\ &- \frac{5}{2\pi\mathrm{i}}\int_\rho^\infty\left(\hat{U}^{(N)}(t) - \mathrm{e}^{\mathrm{i}\pi/2}\frac{3^{1/4}}{2\sqrt{\pi}}\, t^{5N-9/4}\mathrm{e}^{-(8\sqrt{3}/5)t^{5/2}}\right)\mathrm{d}t + r_N^{(1)}(\rho) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{5\cdot 3^{1/4}}{4\pi^{3/2}}\int_0^\infty t^{5N-9/4}\mathrm{e}^{-(8\sqrt{3}/5)t^{5/2}}\,\mathrm{d}t + \frac{5\cdot 3^{1/4}}{4\pi^{3/2}}\int_0^\rho t^{5N-9/4}\mathrm{e}^{-(8\sqrt{3}/5)t^{5/2}}\,\mathrm{d}t \\
&\quad - \frac{5}{2\pi\mathrm{i}}\int_\rho^\infty \left(\hat{U}^{(N)}(t) - \mathrm{e}^{\mathrm{i}\pi/2}\frac{3^{1/4}}{2\sqrt{\pi}}\,t^{5N-9/4}\mathrm{e}^{-(8\sqrt{3}/5)t^{5/2}}\right)dt \\
&\quad + r_N^{(1)}(\rho) \equiv I_1 + I_2 + I_3 + +r_N^{(1)}(\rho).
\end{aligned}
\tag{2.34}
$$

The first integral, I_1, in the last equation reduces to the Gamma-function integral,

$$
\begin{aligned}
I_1 &= -\frac{5\cdot 3^{1/4}}{4\pi^{3/2}}\int_0^\infty t^{5N-9/4}\mathrm{e}^{-(8\sqrt{3}/5)t^{5/2}}\,\mathrm{d}t = -\frac{3^{1/4}}{2\pi^{3/2}}\left(\frac{8\sqrt{3}}{5}\right)^{-2N+1/2}\int_0^\infty s^{2N-3/2}\mathrm{e}^{-s}\,\mathrm{d}s \\
&= -\frac{3^{1/4}}{2\pi^{3/2}}\left(\frac{8\sqrt{3}}{5}\right)^{-2N+1/2}\Gamma\left(2N-\frac{1}{2}\right).
\end{aligned}
\tag{2.35}
$$

For the second integral, I_2, we have

$$
\begin{aligned}
|I_2| &= \frac{5\cdot 3^{1/4}}{4\pi^{3/2}}\left|\int_0^\rho t^{5N-9/4}\mathrm{e}^{-(8\sqrt{3}/5)t^{5/2}}\,\mathrm{d}t\right| \le \frac{5\cdot 3^{1/4}}{4\pi^{3/2}}\int_0^\rho t^{5N-9/4}\,\mathrm{d}t \\
&= \frac{5\cdot 3^{1/4}}{4\pi^{3/2}}\int_0^\rho t^{5N-9/4}\,\mathrm{d}t = \frac{3^{1/4}}{4\pi^{3/2}(N-1/4)}\rho^{5N-5/4} < C^{(2)}\frac{\rho^{5N}}{N}.
\end{aligned}
\tag{2.36}
$$

For the third integral, I_3, we use the asymptotics (2.32),

$$
|I_3| < C\frac{5\cdot 3^{1/4}}{4\pi\sqrt{\pi}}\int_\rho^\infty t^{5N-19/4}\mathrm{e}^{-(8\sqrt{3}/5)t^{5/2}}\,\mathrm{d}t < C^{(3)}\Gamma\left(2N-\frac{3}{2}\right) + C^{(4)}\frac{\rho^{5N}}{N},
\tag{2.37}
$$

where the constants $C^{(j)}$, $j = 2, 3, 4$ are positive constants which do not depend on N. Equation (2.35) and estimates (2.36), (2.37), and (2.30) mean that

$$
a_N = -\frac{3^{1/4}}{2\pi^{3/2}}\left(\frac{8\sqrt{3}}{5}\right)^{-2N+1/2}\Gamma\left(2N-\frac{1}{2}\right)(1+\mathcal{O}(N^{-1})) + \mathcal{O}(\rho^{5N}),
\tag{2.38}
$$

as $N \to \infty$. This, in turn, implies the following expression for the leading term of the large N asymptotics of the coefficient a_N.

$$a_N == -\frac{3^{1/4}}{2\pi^{3/2}}\left(\frac{8\sqrt{3}}{5}\right)^{-2N+1/2} \Gamma\left(2N-\frac{1}{2}\right)(1+\mathcal{O}(N^{-1})), \quad N \longrightarrow \infty. \tag{2.39}$$

Equation (2.39) provides us with the leading term of the large n asymptotics of $a_n \equiv u_{n,0}$. In order to reproduce the whole series (1.4) we note that the Riemann–Hilbert analysis of the tritronquée solutions yields in fact the full asymptotic series expansion for the differences between the tritronquée solutions. This means the following extension of the estimates (2.7):

$$\begin{gathered} u_{k+1}(z) - u_k(z) \sim \mathrm{e}^{\mathrm{i}(\pi/2)(k+1)} \frac{3^{1/4}}{2\sqrt{\pi}} z^{-1/8} \mathrm{e}^{(-1)^{k+1}(8\sqrt{3}/5)z^{5/4}} \left(1+\sum_{n=1}^{\infty} c_n z^{-5n/4}\right), \\ z \longrightarrow \infty, \quad \arg z \in \left(-\frac{2\pi}{5}+\frac{4\pi}{5}k, \frac{2\pi}{5}+\frac{4\pi}{5}k\right), \end{gathered} \tag{2.40}$$

with some coefficients c_n. In its turn, asymptotics (2.40) implies that

$$\hat{U}^{(N)}(t)|_{\arg t=(2\pi/5)k} = \mathrm{e}^{\mathrm{i}(\pi/2)(k+1)} \frac{3^{1/4}}{2\sqrt{\pi}} t^{5N-9/4} \mathrm{e}^{(-1)^{k+1}(8\sqrt{3}/5)t^{5/2}} \left(1+\sum_{n=1}^{m} c_n t^{-5n/2} + \mathcal{O}(t^{-5(m+1)/2})\right). \tag{2.41}$$

Therefore, the integral of $\hat{U}^{(N)}(t)$ along the half-line (ρ, ∞) can be estimated (We omit the routine technical details which are similar to the ones carefully presented in the derivation of (2.39)) as follows:

$$\begin{aligned} &\int_\rho^\infty \hat{U}^{(N)}(t)\,\mathrm{d}t \\ &= -S_1\left(\sum_{n=0}^{m} c_n \int_\rho^\infty \mathrm{e}^{-At^{5/2}} t^{5N-5n/2-9/4}\,\mathrm{d}t + \mathcal{O}\left(\int_\rho^\infty \mathrm{e}^{-At^{5/2}} t^{5N-5(m+1)/2-9/4}\,\mathrm{d}t\right)\right) \\ &= -\frac{2}{5}S_1\left(\sum_{n=0}^{m} c_n A^{-2N+n+1/2}\int_{A\rho^{5/2}}^\infty \mathrm{e}^{-t} t^{2N-n-3/2}\,\mathrm{d}t + \mathcal{O}\left(A^{-2N+m+3/2}\int_{A\rho^{5/2}}^\infty \mathrm{e}^{-t} t^{2N-m-1-3/2}\,\mathrm{d}t\right)\right) \\ &= -\frac{2}{5}S_1\left(\sum_{n=0}^{m} c_n A^{-2N+n+1/2}\Gamma\left(2N-n-\frac{1}{2}\right) + \mathcal{O}\left(A^{-2N+m+3/2}\Gamma\left(2N-m-\frac{3}{2}\right)\right)\right) \\ &\quad + \mathcal{O}(\rho^{5N-5/4}) \end{aligned}$$

$$= -\frac{2}{5} A^{-2N+1/2} \Gamma\left(2N - \frac{1}{2}\right) S_1 \left(1 + \sum_{n=1}^{m} \frac{c_n A^n}{\prod_{k=1}^{n}(2N - 1/2 - k)} + \mathcal{O}\left(\frac{A^{m+1}}{\prod_{k=1}^{m+1}(2N - 1/2 - k)}\right)\right)$$

$$+ \mathcal{O}(\rho^{5N-5/4})$$

$$= -\frac{2}{5} A^{-2N+1/2} \Gamma\left(2N - \frac{1}{2}\right) S_1 \left\{1 + \sum_{n=1}^{m} \frac{c_n A^n}{\prod_{k=1}^{n}(2N - 1/2 - k)} + \mathcal{O}(N^{-m-1})\right\}. \tag{2.42}$$

Here, m is arbitrary but fixed and all the constants in the error terms depend on m and ρ (which is also fixed) only. We have also used our standard notations

$$A = \frac{8\sqrt{3}}{5}, \quad S_1 = -\mathrm{i}\frac{3^{1/4}}{2\sqrt{\pi}}, \quad c_0 = 1.$$

Substituting (2.42) into (2.33), we arrive at the final estimate for a_N,

$$a_N = A^{-2N+1/2} \Gamma\left(2N - \frac{1}{2}\right) \frac{S_1}{\pi \mathrm{i}} \left\{1 + \sum_{n=1}^{m} \frac{c_n A^n}{\prod_{k=1}^{n}(2N - 1/2 - k)} + \mathcal{O}(N^{-m-1})\right\}. \tag{2.43}$$

To complete the proof of the 0-instanton asymptotics (1.4), we need to identify the coefficients c_n as the 1-instanton coefficients $u_{n,1}$. To this end, it is enough to note that the difference, $v(z) \equiv u_{k+1}(z) - u_k(z)$, of the tritronquée solutions satisfies the linear differential equation,

$$v'' = 6v(u_{k+1} + u_k). \tag{2.44}$$

Since both u_k and u_{k+1} have the same power-series expansion in the relevant sectors, we conclude that the formal power-series solution of Equation (2.44) must coincide with the formal power-series solution of the 1-instanton equation (1.20). Hence the desired equation,

$$c_n = u_{n,1}, \quad \forall n, \tag{2.45}$$

which completes the proof of (1.4).

3 Asymptotics of the 1-instanton of Painlevé I

3.1 A 5-tuple of differential equations

Let us consider the first instanton term $C u_1$ in the formal series (1.9). This term satisfies the linear homogeneous second-order ODE

$$u_1'' = 12 u_0 u_1, \tag{3.1}$$

where u_0 is the 0-instanton term in the trans-series expansion (1.9). This suggests the problem of studying the linear equation

$$v'' = \gamma u v, \tag{3.2}$$

where u is one of the tritronquée solutions of the first Painlevé equation (1.1) and γ is an arbitrary positive constant. Indeed, we have five *different* Schrödinger equations with meromorphic potentials γu_j having the *same* formal power expansion (2.1) in the relevant sectors, see (2.3):

$$v_j'' = \gamma u_j v_j, \quad \gamma u_j \sim \gamma u_f(z) = \gamma z^{1/2} \sum_{n=0}^{\infty} a_n z^{-5n/2}. \tag{3.3}$$

Equation (3.2) has two linearly independent formal solutions,

$$\begin{aligned} &v_f^{\pm} = \mathrm{e}^{\theta^{\pm}} \Sigma^{\pm}, \quad \theta^{\pm} = B^{\pm} z^{5/4} - \frac{1}{8} \ln z, \quad \Sigma^{\pm} = \sum_{n=0}^{\infty} b_n^{\pm} z^{-5n/4}, \\ &B^{\pm} = \pm B = \pm \frac{4}{5} \sqrt{\gamma}, \quad b_0^{\pm} = 1, \\ &b_n^{\pm} = \frac{1}{2n} \left\{ \frac{b_{n-1}^{\pm}}{B^{\pm}} \left(n - \frac{1}{10} \right) \left(n - \frac{9}{10} \right) - B^{\pm} \sum_{m=1}^{[(n+1/2)]} a_m b_{n+1-2m}^{\pm} \right\}, \quad n = 1, 2, \ldots. \end{aligned} \tag{3.4}$$

To simplify our notation, in this section, we use the symbol b_n^- instead of $u_{n,1}$.

Since we have five equations with meromorphic potentials, we have five pairs of solutions with the asymptotic expansions (3.4) as $z \longrightarrow \infty$. Each pair has this asymptotic expansion in the sector where the corresponding potential has the power-like asymptotics (2.1). The sectors are depicted in Figure 1. Denote the solutions corresponding to the potential $u_j(z)$ by the symbol $v_j^{\pm}(z)$. We have,

$$v_j^{\pm}(z) \sim v_f^{\pm}(z), \quad z \longrightarrow \infty, \quad \arg z \in \left(-\frac{6\pi}{5} + \frac{4\pi}{5} j, \frac{2\pi}{5} + \frac{4\pi}{5} j \right). \tag{3.5}$$

We also observe that

$$v_j^{\pm}(z) = \mathrm{e}^{-\mathrm{i}\pi/10 j} v_0^{\sigma_j(\pm)} (\mathrm{e}^{-\mathrm{i}(4\pi/5) j} z), \quad \sigma_j(\pm) = \begin{cases} \pm, & j \text{ is even}, \\ \mp, & j \text{ is odd}, \end{cases} \tag{3.6}$$

and

$$v^{\pm}_{j+20}(z) = v^{\pm}_{j}(z). \tag{3.7}$$

Similar to convention (2.10) concerning the subscript k in $u_k(x)$, we shall assume that

$$j \in \mathbb{Z}, \quad \bmod 20, \tag{3.8}$$

in the notation $v^{\pm}_{j}(z)$, unless j is particularly specified, as in (3.19).

3.2 Gluing the solutions together

We observe that within the common sectors of the same asymptotic behavior, the solutions $v^{\pm}_{j}(z)$ and $v^{\pm}_{j-1}(z)$ have identical asymptotics $v^{\pm}_{f}(z)$ in all orders. Thus, it is natural to ask: what is the difference between these two solutions of *different* equations? Let us introduce the differences

$$w^{\pm}_{j}(z) = v^{\pm}_{j+1}(z) - v^{\pm}_{j}(z). \tag{3.9}$$

Because of (3.6), these differences are related to each other by the following rotational symmetry:

$$w^{\pm}_{j}(z) = \mathrm{e}^{-\mathrm{i}(\pi/10)j} w_0^{\sigma_j(\pm)}(\mathrm{e}^{-\mathrm{i}(4\pi/5)j} z). \tag{3.10}$$

In addition, using (3.3), $w^{\pm}_{j}(z)$ satisfies the nonhomogeneous linear ODE,

$$(w^{\pm}_{j-1})_{zz} = \gamma u_j w^{\pm}_{j-1} + \gamma (u_j - u_{j-1}) v^{\pm}_{j-1}. \tag{3.11}$$

The homogeneous part of this equation is the same as in (3.3) In the relevant sector, the nonhomogeneity contains an exponentially small factor given in (2.7):

$$\begin{gathered} z \longrightarrow \infty, \quad \arg z \in \left(-\frac{6\pi}{5} + \frac{4\pi}{5} j, -\frac{2\pi}{5} + \frac{4\pi}{5} j\right): \\ u_j(z) - u_{j-1}(z) = \mathrm{e}^{\mathrm{i}(\pi/2)j} \frac{3^{1/4}}{2\sqrt{\pi}} \mathrm{e}^{\theta_j}(1 + \mathcal{O}(z^{-5/4})), \\ \theta_j = \alpha_j z^{5/4} - \frac{1}{8} \ln z, \quad \alpha_j = (-1)^j A, \quad A = \frac{8\sqrt{3}}{5}. \end{gathered} \tag{3.12}$$

The general solution to (3.11) is given by the integral formula,

$$w^{\pm}_{j-1}(z) = c^{+}v^{+}_{j}(z) + c^{-}v^{-}_{j} - \frac{1}{2\sqrt{\gamma}}\int_{z_0}^{z}(v^{+}_{j}(x)v^{-}_{j}(z) - v^{+}_{j}(z)v^{-}_{j}(x))\gamma(u_j(x) - u_{j-1}(x))v^{\pm}_{j-1}(x)\,\mathrm{d}x, \tag{3.13}$$

where c^{+}, c^{-}, and z_0 are arbitrary constants. However, our solution (3.9) is *not* general as being a difference between two functions with the same asymptotic expansion in certain sector. Furthermore, in the sector $\arg z \in (-\frac{6\pi}{5} + \frac{4\pi}{5}j, -\frac{2\pi}{5} + \frac{4\pi}{5}j)$, see (3.12), for j odd, the solution $v^{+}_{j}(z)$ is dominant and $v^{-}_{j}(z)$ is recessive, while for j even, $v^{+}_{j}(z)$ is recessive and $v^{-}_{j}(z)$ is dominant.Thus, $w^{\pm}_{j-1}(z)$ admits the following representations:

$$\begin{aligned}
&j\ \text{odd}: v^{+}_{j}(z)\ \text{dominant},\ v^{-}_{j}(z)\ \text{recessive as}\ \arg z \in \left(-\frac{6\pi}{5} + \frac{4\pi}{5}j, -\frac{2\pi}{5} + \frac{4\pi}{5}j\right):\\
&w^{+}_{j-1}(z) = c_0 v^{-}_{j}(z) - \frac{1}{2}\sqrt{\gamma}v^{-}_{j}(z)\int_{\mathrm{e}^{\mathrm{i}(4\pi/5)(j-1)}z_0}^{z}(u_j(x) - u_{j-1}(x))v^{+}_{j}(x)v^{+}_{j-1}(x)\,\mathrm{d}x\\
&\qquad + \frac{1}{2}\sqrt{\gamma}v^{+}_{j}(z)\int_{\mathrm{e}^{\mathrm{i}(4\pi/5)(j-1)}\infty}^{z}(u_j(x) - u_{j-1}(x))v^{-}_{j}(x)v^{+}_{j-1}(x)\,\mathrm{d}x,\\
&w^{-}_{j-1}(z) = -\frac{1}{2}\sqrt{\gamma}\int_{\mathrm{e}^{\mathrm{i}(4\pi/5)(j-1)}\infty}^{z}(v^{+}_{j}(x)v^{-}_{j}(z) - v^{+}_{j}(z)v^{-}_{j}(x))(u_j(x) - u_{j-1}(x))v^{-}_{j-1}(x)\,\mathrm{d}x,
\end{aligned} \tag{3.14a}$$

$$\begin{aligned}
&j\ \text{even}: v^{+}_{j}(z)\ \text{recessive},\ v^{-}_{j}(z)\ \text{dominant as}\ \arg z \in \left(-\frac{6\pi}{5} + \frac{4\pi}{5}j, -\frac{2\pi}{5} + \frac{4\pi}{5}j\right):\\
&w^{+}_{j-1}(z) = -\frac{1}{2}\sqrt{\gamma}\int_{\mathrm{e}^{\mathrm{i}(4\pi/5)(j-1)}\infty}^{z}(v^{+}_{j}(x)v^{-}_{j}(z) - v^{+}_{j}(z)v^{-}_{j}(x))(u_j(x) - u_{j-1}(x))v^{+}_{j-1}(x)\,\mathrm{d}x,\\
&w^{-}_{j-1}(z) = c_0 v^{+}_{j}(z) - \frac{1}{2}\sqrt{\gamma}v^{-}_{j}(z)\int_{\mathrm{e}^{\mathrm{i}(4\pi/5)(j-1)}\infty}^{z}(u_j(x) - u_{j-1}(x))v^{+}_{j}(x)v^{-}_{j-1}(x)\,\mathrm{d}x\\
&\qquad + \frac{1}{2}\sqrt{\gamma}v^{+}_{j}(z)\int_{\mathrm{e}^{\mathrm{i}(4\pi/5)(j-1)}z_0}^{z}(u_j(x) - u_{j-1}(x))v^{-}_{j}(x)v^{-}_{j-1}(x)\,\mathrm{d}x,
\end{aligned} \tag{3.14b}$$

where $\mathrm{e}^{\mathrm{i}(4\pi/5)(j-1)}z_0$ is a finite point within the indicated sector and c_0 is a Stokes constant which both can not be determined immediately.

When $z \longrightarrow \infty$ as $\arg z \in (-\frac{6\pi}{5} + \frac{4\pi}{5}j, -\frac{2\pi}{5} + \frac{4\pi}{5}j)$, the leading contribution to the asymptotic behavior of $w^{\pm}_{j-1}(z)$ is given by z-end point of the integral term and possibly by the nonintegral term. By standard arguments,

$$j\ \text{odd}: v^{+}_{j}(z)\ \text{dominant},\ v^{-}_{j}(z)\ \text{recessive as}\ \arg z \in \left(-\frac{6\pi}{5} + \frac{4\pi}{5}j, -\frac{2\pi}{5} + \frac{4\pi}{5}j\right):$$

$$w^{+}_{j-1}(z) = \begin{cases} -\mathrm{e}^{\mathrm{i}(\pi/2)j} \dfrac{3^{1/4}}{5\sqrt{\pi}} \dfrac{2B\sqrt{\gamma}}{A(-A+2B)} \mathrm{e}^{(-A+B)z^{5/4}} z^{-3/4}(1+\mathcal{O}(z^{-5/4})), & B > A/2, \\ -\mathrm{e}^{\mathrm{i}(\pi/2)j} \dfrac{2\cdot 3^{1/4}}{5\sqrt{\pi}} \sqrt{\gamma}\mathrm{e}^{-Bz^{5/4}} z^{1/2}(1+\mathcal{O}(z^{-5/8})), & B = A/2, \\ c_1 \mathrm{e}^{-Bz^{5/4}} z^{-1/8}(1+\mathcal{O}(z^{-5/4})), & 0 < B < A/2, \end{cases}$$

$$w^{-}_{j-1}(z) = \mathrm{e}^{\mathrm{i}(\pi/2)j} \frac{3^{1/4}}{5\sqrt{\pi}} \frac{2B\sqrt{\gamma}}{A(A+2B)} \mathrm{e}^{-(A+B)z^{5/4}} z^{-3/4}(1+\mathcal{O}(z^{-5/4})), \tag{3.15a}$$

j even : $v^{+}_{j}(z)$ recessive, $v^{-}_{j}(z)$ dominant as $\arg z \in \left(-\frac{6\pi}{5} + \frac{4\pi}{5}j, -\frac{2\pi}{5} + \frac{4\pi}{5}j\right)$:

$$w^{+}_{j-1}(z) = \mathrm{e}^{\mathrm{i}(\pi/2)j} \frac{3^{1/4}}{5\sqrt{\pi}} \frac{2B\sqrt{\gamma}}{A(A+2B)} \mathrm{e}^{(A+B)z^{5/4}} z^{-3/4}(1+\mathcal{O}(z^{-5/4}))$$

$$w^{-}_{j-1}(z) = \begin{cases} \mathrm{e}^{\mathrm{i}(\pi/2)j} \dfrac{3^{1/4}}{5\sqrt{\pi}} \dfrac{2B\sqrt{\gamma}}{A(A-2B)} \mathrm{e}^{(A-B)z^{5/4}} z^{-3/4}(1+\mathcal{O}(z^{-5/4})), & B > A/2, \\ \mathrm{e}^{\mathrm{i}(\pi/2)j} \dfrac{2\cdot 3^{1/4}}{5\sqrt{\pi}} \sqrt{\gamma}\mathrm{e}^{Bz^{5/4}} z^{1/2}(1+\mathcal{O}(z^{-5/8})), & B = A/2, \\ c_1 \mathrm{e}^{Bz^{5/4}} z^{-1/8}(1+\mathcal{O}(z^{-5/4})), & 0 < B < A/2, \end{cases} \tag{3.15b}$$

where c_1 is an unknown Stokes constant.

Taking into account the presence of the factor z^{-8} in the asymptotics of the functions $v^{\pm}_{j}(\xi)$, we define auxiliary functions $\hat{v}^{\pm}_{j}(\xi)$ by the relations (cf. (2.13)),

$$z = \xi^{8}, \quad \hat{v}^{\pm}_{j}(\xi) = v^{\pm}_{j}(z)\mathrm{e}^{-\theta^{\pm}(z)}, \tag{3.16}$$

so that (see (3.5))

$$\hat{v}^{\pm}_{j}(\xi) \sim \Sigma^{\pm}(\xi^{8}) = \sum_{n=0}^{\infty} b^{\pm}_{n}\xi^{-10n}, \quad \xi \longrightarrow \infty, \quad \arg\xi \in \left(-\frac{3\pi}{20} + \frac{\pi}{10}j, \frac{\pi}{20} + \frac{\pi}{10}j\right). \tag{3.17}$$

Using (3.6), we observe that

$$\hat{v}^{\pm}_{j}(\xi) = \mathrm{e}^{-\mathrm{i}(\pi/10)j}\hat{v}^{\sigma_j(\pm)}_{0}(\mathrm{e}^{-\mathrm{i}(\pi/10)j}\xi), \quad \hat{v}^{\pm}_{j+20}(\xi) = \hat{v}^{\pm}_{j}(\xi). \tag{3.18}$$

Let us introduce the sectorially meromorphic function $\hat{V}^{(N)}(\xi)$ discontinuous across the rays $\arg\xi = \frac{\pi}{10}j$, $j = -9, -8, \dots, 9, 10$:

$$\hat{V}^{(N)}(\xi) = \hat{v}^{-}_{j}(\xi)\xi^{10N-1} - P^{-}_{10N-1}(\xi), \quad \arg\xi \in \left(-\frac{\pi}{10} + \frac{\pi}{10}j, \frac{\pi}{10}j\right), \quad j = -9, -8, \dots, 9, 10, \tag{3.19}$$

where polynomial $P^{-}_{10N-1}(\xi)$ of degree $10N-1$ is defined by

$$P^{-}_{10N-1}(\xi) := \sum_{n=0}^{N-1} b^{-}_{n}\xi^{10(N-n)-1}. \tag{3.20}$$

According to (3.17), this function has the uniform asymptotics at infinity,

$$V^{(N)}(\xi) = b^{-}_{n}\xi^{-1} + \mathcal{O}_N(\xi^{-11}), \quad \xi \longrightarrow \infty, \tag{3.21}$$

and, due to (3.9) and (3.16), it has the following jumps across the rays $\arg\xi = \frac{\pi}{10}(j-1)$, $j = -9, -8, \ldots, 9, 10$, oriented towards infinity:

$$\begin{aligned}\arg\xi = \frac{\pi}{10}(j-1):\quad & \hat{V}^{(N)}_{+}(\xi) - \hat{V}^{(N)}_{-}(\xi) = (\hat{v}^{-}_{j}(\xi) - \hat{v}^{-}_{j-1}(\xi))\xi^{10N-1}\\ &= (v^{-}_{j}(z) - v^{-}_{j-1}(z))\mathrm{e}^{-\theta^{-}(z)}\xi^{10N-1} = w^{-}_{j-1}(z)\mathrm{e}^{-\theta^{-}(z)}\xi^{10N-1}, \quad j = -8, -7, \ldots, 11.\end{aligned} \tag{3.22}$$

(We note that $v^{-}_{11}(z) = v^{-}_{-9}(z)$, in virtue of (3.7)). The jumps for even and odd j-s are different. Namely, as $\xi \to \infty$, using (3.15), we find

$$\xi \longrightarrow \infty, \quad \arg\xi = \frac{\pi}{10}(j-1), \quad j \text{ odd}:$$

$$\hat{V}^{(N)}_{+}(\xi) - \hat{V}^{(N)}_{-}(\xi) = \mathrm{e}^{\mathrm{i}(\pi/2)j}\frac{3^{1/4}}{5\sqrt{\pi}}\frac{2B\sqrt{\gamma}}{A(A+2B)}\mathrm{e}^{-A\xi^{10}}\xi^{10N-6}(1+\mathcal{O}(\xi^{-10})), \tag{3.23a}$$

$$\xi \longrightarrow \infty, \quad \arg\xi = \frac{\pi}{10}(j-1), \quad j \text{ even}:$$

$$\hat{V}^{(N)}_{+}(\xi) - \hat{V}^{(N)}_{-}(\xi) = \begin{cases} \mathrm{e}^{\mathrm{i}(\pi/2)j}\dfrac{3^{1/4}}{5\sqrt{\pi}}\dfrac{2B\sqrt{\gamma}}{A(A-2B)}\mathrm{e}^{A\xi^{10}}\xi^{10N-6}(1+\mathcal{O}(\xi^{-10})), & B > A/2,\\ \mathrm{e}^{\mathrm{i}(\pi/2)j}\dfrac{2\cdot 3^{1/4}}{5\sqrt{\pi}}\sqrt{\gamma}\mathrm{e}^{A\xi^{10}}\xi^{10N+4}(1+\mathcal{O}(\xi^{-5})), & B = A/2,\\ c_1\mathrm{e}^{2B\xi^{10}}\xi^{10N-1}(1+\mathcal{O}(\xi^{-10})), & 0 < B < A/2.\end{cases} \tag{3.23b}$$

Remark 3.1. In this section, as well as in the rest of the paper, we skip the explanation of the standard meaning of the symbols $\mathcal{O}(\ldots)$ and $\mathcal{O}_N(\ldots)$. This meaning has been explained in Section 2 (see, e.g. (2.9), (2.15), (2.22), and (2.24)). We also omit, when performing asymptotic calculations, the routine technical details. They are similar to those which were carefully presented in Section 2. □

3.3 An integral formula for the 1-instanton coefficients and their asymptotic expansion

Due to (3.21), the 1-instanton coefficient b_N^-, whose N-large asymptotics we are looking for, is the coefficient at the term $1/\xi$ of the asymptotic expansion of $V^{(N)}(\xi)$ at $\xi=\infty$. Therefore,

$$\frac{1}{2\pi\mathrm{i}}\oint_{|\xi|=R\gg 1} V^{(N)}(\xi)\,\mathrm{d}\xi = b_n^- + \mathcal{O}_N(R^{-10}). \tag{3.24}$$

Similarly to the 0-instanton case, collapsing this circular path of integration to a circle of a smaller radius $|\xi|=r$ containing all the singularities or branch points of $\hat{V}^{(N)}(\xi)$ (the latter can appear at the poles of $u_j(\xi^8)$ only), we find

$$\begin{aligned}
b_N^- &= \frac{1}{2\pi\mathrm{i}}\oint_{|\xi|=R\gg 1} V^{(N)}(\xi)\,\mathrm{d}\xi + \mathcal{O}_N(R^{-10})\\
&= -\frac{1}{2\pi\mathrm{i}}\sum_{j=-9}^{10}\int_{r\mathrm{e}^{\mathrm{i}(\pi/10)(j-1)}}^{R\mathrm{e}^{\mathrm{i}(\pi/10)(j-1)}} w_{j-1}^-(z)\mathrm{e}^{-\theta^-(z)}\xi^{10N-1}\,\mathrm{d}\xi + \frac{1}{2\pi\mathrm{i}}\oint_{|\xi|=r} V^{(N)}(\xi)\,\mathrm{d}\xi + \mathcal{O}_N(R^{-10})\\
&= -\frac{1}{2\pi\mathrm{i}}\sum_{j=-9}^{10}\int_{r\mathrm{e}^{\mathrm{i}(\pi/10)(j-1)}}^{R\mathrm{e}^{\mathrm{i}(\pi/10)(j-1)}} w_{j-1}^-(z)\mathrm{e}^{-\theta^-(z)}\xi^{10N-1}\,\mathrm{d}\xi\\
&\quad + \frac{1}{2\pi\mathrm{i}}\oint_{|\xi|=r} (V^{(N)}(\xi)+P_{10N-1}^-(\xi))\,\mathrm{d}\xi + \mathcal{O}_N(R^{-10})\\
&= -\frac{1}{2\pi\mathrm{i}}\sum_{j=-9}^{10}\int_{r\mathrm{e}^{\mathrm{i}(\pi/10)(j-1)}}^{R\mathrm{e}^{\mathrm{i}(\pi/10)(j-1)}} \mathrm{e}^{-\mathrm{i}(\pi/10)(j-1)} w_0^{\sigma_{j-1}(-)}(\mathrm{e}^{-\mathrm{i}(4\pi/5)(j-1)}z)\mathrm{e}^{-\theta^-(z)}\xi^{10N-1}\,\mathrm{d}\xi\\
&\quad + \mathcal{O}(r^{10N}) + \mathcal{O}_N(R^{-10})\\
&= -\frac{1}{2\pi\mathrm{i}}\sum_{k=-5}^{4}\int_{r\mathrm{e}^{\mathrm{i}(\pi/5)k}}^{R\mathrm{e}^{\mathrm{i}(\pi/5)k}} \mathrm{e}^{-\mathrm{i}(\pi/5)k} w_0^-((\mathrm{e}^{-\mathrm{i}(\pi/5)k}\xi)^8)\mathrm{e}^{B\xi^{10}}\xi^{10N}\,\mathrm{d}\xi\\
&\quad - \frac{1}{2\pi\mathrm{i}}\sum_{k=-5}^{4}\int_{r\mathrm{e}^{\mathrm{i}(\pi/10)(2k+1)}}^{R\mathrm{e}^{\mathrm{i}(\pi/10)(2k+1)}} \mathrm{e}^{-(\mathrm{i}/\pi 10)(2k+1)} w_0^+((\mathrm{e}^{-\mathrm{i}(\pi/10)(2k+1)}\xi)^8)\mathrm{e}^{B\xi^{10}}\xi^{10N}\,\mathrm{d}\xi\\
&\quad + \mathcal{O}(r^{10N}) + \mathcal{O}_N(R^{-10})\\
&= -\frac{5}{\pi\mathrm{i}}\int_r^R w_0^-(\xi^8)\mathrm{e}^{B\xi^{10}}\xi^{10N}\,\mathrm{d}\xi - \frac{5}{\pi\mathrm{i}}(-1)^N\int_r^R w_0^+(\xi^8)\mathrm{e}^{-B\xi^{10}}\xi^{10N}\,\mathrm{d}\xi + \mathcal{O}(r^{10N}) + \mathcal{O}_N(R^{-10}).
\end{aligned} \tag{3.25}$$

Here, we used that

$$|V^{(N)}(\xi)+P_{10N-1}^-(\xi)|_{|\xi|=r} < Cr^{10N-1},$$

where the constant C does not depend on N. It is not difficult to find N-large asymptotics of the coefficients b_N^-. Similar to the 0-instanton case considered in Section 2, we first send $R \to \infty$ in the last equation of (3.25) and then, using (3.15a), analyze the large N behavior of the resulting integrals:

$$b_N^- = \begin{cases} \dfrac{4 \cdot 3^{1/4}}{25\pi^{3/2}} \dfrac{A(1+(-1)^N) - 2B(1-(-1)^N)}{4B^2 - A^2} \gamma A^{-N-1/2} \Gamma\left(N - \dfrac{1}{2}\right)(1 + \mathcal{O}(N^{-1})), & \gamma > 3, \\ (-1)^N \dfrac{3^{1/4}}{5\pi^{3/2}} \sqrt{\gamma}(2B)^{-N-1/2} \Gamma\left(N + \dfrac{1}{2}\right)(1 + \mathcal{O}(N^{-1/2})), & \gamma = 3, \\ i(-1)^N c_1 \dfrac{1}{2\pi}(2B)^{-N}\Gamma(N)(1 + \mathcal{O}(N^{-1})), & 0 < \gamma < 3. \end{cases} \tag{3.26}$$

We recall that $B = \frac{4}{5}\sqrt{\gamma}$ and $A = \frac{8}{5}\sqrt{3}$.

Equation (3.26), in the case $\gamma = 12$, produces the leading term of the asymptotics of the 1-instanton sequence $u_{n,1}$. The proof of the whole series (1.16)–(1.17) is similar to the proof of the 0-instanton asymptotic series (1.4) performed at the end of Section 2.

Remark 3.2. Note, that for $\gamma > 3$, the unknown Stokes constant c_1 does not contribute to the leading term of the large N asymptotics of b_N^-; indeed, it appears in the exponentially small terms only. For $\gamma = 3$, this contribution appears in the subleading term of order $N^{-1/2}$. However, for $0 < \gamma < 3$, the asymptotics of b_N^- is simply proportional to c_1. This is also confirmed by numerical computations. □

3.4 Tritronquée solutions and the induced Stokes' phenomenon

The analysis performed in the previous subsections gives rise to the following observations concerning the Schrödinger equation:

$$v'' = \gamma u v. \tag{3.27}$$

Let us take two different tritronquée Painlevé I functions, say the functions $u_0(z)$ and $u_1(z)$, as the potentials $u(z)$ in Equation (3.27). The corresponding solutions, $v_0^+(z)$ and $v_1^+(z)$, though solutions to *different* equations, would have the *same* asymptotics in all orders in the common sector, $-2\pi/5 < \arg z < 2\pi/5$. The same is true for the pair $v_0^-(z)$ and $v_1^-(z)$. Hence, the possibility arises of evaluating the exponentially small differences, $v_1^+(z) - v_1^+(z)$ and $v_1^-(z) - v_1^-(z)$. In other words, the existence of the tritronquée Painlevé transcendents allows us to introduce the notion of the Stokes' phenomenon for the collection of the solutions to the family of linear equations (3.27) consisting of five equations generated by five tritronquée functions as potentials $u(z)$. We shall call this phenomenon the *induced Stokes' phenomenon* or, more lengthy, the *induced Stokes' phenomenon generated by the first Painlevé quasi-linear Stokes' phenomenon*. It is worth noting that no single equation (3.27) with $u(z) = u_k(z)$ generates a meaningful Stokes phenomenon in the set of its solutions. Indeed, the structure of the solutions to Equation (3.27) in the complementary sector, where the potential has infinitely many poles (see Figure 1), is extremely difficult to describe. We need five different potentials in (3.27) to cover the neighborhood of infinity by the sectors with the regular behavior of both potentials and the solutions.

There is a threshold value of the parameter γ in (3.27), namely, $\gamma = 3$. When $\gamma > 3$, the induced Stokes' phenomenon generated by the first Painlevé quasi-linear Stokes phenomenon is completely described by the latter. Indeed, the origin of the coefficient $3^{1/4}/2\sqrt{\pi}$ in formulae (3.15a) and (3.15b) is the Stokes' constant

$$S_1 = -\mathrm{i}\frac{3^{1/4}}{2\sqrt{\pi}},$$

which is featuring in the quasi-linear Stokes' relations (2.7) for the tritronquée solutions $u_k(z)$. When $\gamma < 3$, the quasi-linear Stokes' phenomenon for potentials of equation (3.27) does not control the induced Stokes' phenomenon for its solutions. The intrinsic Stokes' constant—the constant c_1, begins to play a dominant role.

There are at least two values of the parameter γ which are less than 3 but for which we believe an explicit description of the induced Stokes' phenomenon is possible. These values are (Painlevé I equation (1.1) provides the linear equation (3.2) with the potential $u(z)$. We note that the value of the parameter γ in Equation (3.2) cannot be changed via the scaling of the variables in the Painlevé equation without violation of the chosen form of the latter. The values (3.28) correspond to form (1.1) of the equation PI chosen in this paper.)

$$\gamma = 2 \quad \text{and} \quad \gamma = \tfrac{3}{4}. \tag{3.28}$$

In the case of $\gamma = 2$, the two linear independent solutions of equation (3.27) admit the following representation,

$$v^{+}(z) = \Psi_{21}(0, -z6^{1/5}), \quad v^{-}(z) = \Psi_{22}(0, -z6^{1/5}), \tag{3.29}$$

where $\Psi(\lambda, x)$ is the 2×2 matrix solution of the Riemann–Hilbert problem associated with the first Painlevé equation. This Riemann–Hilbert problem is used in [22] for evaluation of the Stokes constant S_1. Equation (3.29) allow us to use the same Riemann–Hilbert problem to evaluate the Stokes parameter associated with the functions $v^{\pm}(z)$.

The Stokes constant corresponding to the case $\gamma = \frac{3}{4}$ was conjectured in [17] on the basis of the relation of equation (3.27) with $\gamma = \frac{3}{4}$ to the symmetric quartic matrix model studied by Brézin–Neuberger [2] and Harris–Martinec [19]. This relation suggests the existence of an alternative Riemann–Hilbert representation of the first Painlevé transcendent which, through the formulae similar to (3.29) would generate solutions of equation (3.27) with $\gamma = 3/4$. We expect this alternative Riemann–Hilbert problem can be deduced with the help of the relevant double scaling limit of the Lax pair for the skew-orthogonal polynomials associated with the symmetric quartic matrix model [2].

4 Trans-series

4.1 Alien derivatives and resurgence equations

An alternative method to obtain the asymptotic behavior of the instanton solutions to the Painlevé I equation is based on the ideas of resurgence introduced by Écalle. In this approach, one first constructs the *trans-series* solution to the differential equation and computes its *alien derivatives*. Using these, one can easily deduce the asymptotics by using contour deformation arguments. This approach has been used in [7–9], see also [17, Section 4], for an application to a first-order differential equation of the Riccati type. In the case of Painlevé I we deal with a *resonant* equation, that is, there is an integer linear combination of its eigenvalues which is null. In this case, the method of resurgence is slightly more subtle. In particular, the relevant trans-series solution involves logarithms, as we will see.

As we mentioned in Section 1, the instanton solutions to Painlevé I appear as trans-series solutions, see (1.9). In order to understand their asymptotics we need, however, a more general trans-series solution, which is a formal, *two*-parameter series of the form

$$u(z, C_1, C_2) = \sum_{n,m \geq 0} C_1^n C_2^m u_{n|m}(z). \tag{4.1}$$

The $u_{n|m}(z)$ have the structure

$$u_{n|m}(z) = z^{1/2} \mathrm{e}^{-(n-m)Az^{5/4}} \phi_{n|m}(z), \tag{4.2}$$

where $\phi_{n|m}(z)$ do not contain exponentials $\mathrm{e}^{\pm Az^{5/4}}$. When $C_2 = 0$ and $C_1 = C$ we recover the trans-series solution (1.9), therefore

$$u_{n|0}(z) = u_n(z). \tag{4.3}$$

Note that the more general trans-series (4.1) is not proper (in the sense explained in Section 1), since for any given direction in the complex plane, there is an infinite number of terms in (4.1) involving exponentials which grow big as $z \to \infty$ along that direction. Therefore, the trans-series (4.1) is more general than those considered in for example [4].

If we substitute (4.1) into (1.1) and collect the coefficients of $C_1^n C_2^m$, we find that the $u_{n|m}(z)$ obey a set of coupled, inhomogeneous linear ODEs

$$u''_{n|m} - 6 \sum_{k,l \geq 0} u_{k|l} u_{n-k|m-l} = 0 \tag{4.4}$$

generalizing (1.10). In the following, it will be convenient to introduce the variable

$$x = z^{5/4}. \tag{4.5}$$

We now compute the alien derivatives of these solutions. The alien derivative Δ_ω of a formal power series $\phi(x)$ was introduced by Écalle [12–14]. To obtain $\Delta_\omega \phi$, one essentially computes the discontinuity of the Borel transform of $\phi(x)$ and $\hat{\phi}(\zeta)$, at the cut in the Borel plane starting at $\zeta = \omega A$. This leads under suitable assumptions to a series in $\mathbb{C}\{\zeta\}$, whose inverse Borel transform is the alien derivative $\Delta_\omega \phi$, see also [3, 26] for more precise definitions. A crucial property is that the pointed alien derivative

$$\dot{\Delta}_\omega = \mathrm{e}^{\omega A x} \Delta_\omega \tag{4.6}$$

commutes with the standard derivative. This makes possible to relate alien derivatives to trans-series solutions. In our case, if we apply the pointed alien derivative to the Painlevé I equation we obtain the linear, second-order ODE

$$-\frac{1}{6} \frac{\mathrm{d}^2}{\mathrm{d}z^2} (\dot{\Delta}_\omega u(z, C_1, C_2)) + 2u(z, C_1, C_2) \dot{\Delta}_\omega u(z, C_1, C_2) = 0. \tag{4.7}$$

This equation has two linearly independent solutions:

$$\frac{\partial u(z, C_1, C_2)}{\partial C_1}, \qquad \frac{\partial u(z, C_1, C_2)}{\partial C_2}, \tag{4.8}$$

and we conclude that

$$\dot{\Delta}_{\omega} u(z, C_1, C_2) = a_{\omega}(C_1, C_2)\frac{\partial u(z, C_1, C_2)}{\partial C_1} + b_{\omega}(C_1, C_2)\frac{\partial u(z, C_1, C_2)}{\partial C_2}, \tag{4.9}$$

where $a_{\omega}(C_1, C_2)$ and $b_{\omega}(C_1, C_2)$ are in principle formal power series in C_1 and C_2 but they are independent of z. This type of equation, relating the pointed alien derivative to trans-series solutions, is called in Écalle's theory the *bridge equation*; see [3, 26] and particularly [18] for an example of a second-order difference equation. In order to understand the asymptotics, we are particularly interested in the cases $\omega = \pm 1$. Let us first analyze the case $\omega = 1$. We find

$$\begin{aligned} \sum_{n,m\geq 0} \mathrm{e}^{-(n+1-m)Ax} C_1^n C_2^m \Delta_1 \phi_{n|m} &= a_1(C_1, C_2) \sum_{n\geq 1, m\geq 0} n C_1^{n-1} C_2^m \mathrm{e}^{-(n-m)Ax} \phi_{n|m} \\ &\quad + b_1(C_1, C_2) \sum_{m\geq 1, n\geq 0} m C_1^n C_2^{m-1} \mathrm{e}^{-(n-m)Ax} \phi_{n|m}. \end{aligned} \tag{4.10}$$

Let us first look at the term multiplying e^{-Ax} on both sides. On the left-hand side this corresponds to $n = m$, and we obtain

$$\begin{aligned} \sum_{n\geq 0} (C_1 C_2)^n \Delta_1 \phi_{n|n} &= a_1(C_1, C_2) \sum_{n\geq 0} (n+1)(C_1 C_2)^n \phi_{n+1|m} \\ &\quad + b_1(C_1, C_2) \sum_{m\geq 1} m C_1^{m+1} C_2^{m-1} \phi_{m+1|m}. \end{aligned} \tag{4.11}$$

Since the left-hand side is only a function of $C_1 C_2$, we conclude that

$$a_1(C_1, C_2) = \sum_{k\geq 0} a_{1,k} (C_1 C_2)^k, \quad b_1(C_1, C_2) = C_2^2 \sum_{k\geq 0} b_{1,k} (C_1 C_2)^k. \tag{4.12}$$

If we now equate the different powers of e^{nAx} and C_1 and C_2, we find the equation

$$\Delta_1 \phi_{n|m} = \sum_{k=0}^{\min(n,m)} a_{1,k} (n+1-k) \phi_{n+1-k|m-k} + \sum_{k=0}^{\min(n,m-1)} b_{1,k} (m-1-k) \phi_{n-k|m-1-k}. \tag{4.13}$$

The case $\omega = -1$ is very similar. We now have that

$$a_{-1}(C_1, C_2) = C_1^2 \sum_{k\geq 0} a_{-1,k} (C_1 C_2)^k, \quad b_{-1}(C_1, C_2) = \sum_{k\geq 0} b_{1,k} (C_1 C_2)^k \tag{4.14}$$

and

$$\Delta_{-1}\phi_{n|m} = \sum_{k=0}^{\min(n-1,m)} a_{-1,k}(n-1-k)\phi_{n-1-k|m-k} + \sum_{k=0}^{\min(n,m+1)} b_{-1,k}(m+1-k)\phi_{n-k|m+1-k}. \tag{4.15}$$

We are particularly interested in the solutions with $m=0$, corresponding to the instanton solutions of Painlevé I. In this case, the equations for the alien derivatives read (we denote $\phi_{k|0}=\phi_k$)

$$\Delta_1\phi_k = S_1(k+1)\phi_{k+1}, \quad k\geq 0, \tag{4.16}$$

where $S_{\pm 1}=a_{\pm 1,0}$, and

$$\Delta_{-1}\phi_k = S_{-1}(k-1)\phi_{k-1} + \tilde{S}_{-1}\phi_{k|1}, \quad k\geq 0, \tag{4.17}$$

where $\tilde{S}_{-1}=b_{-1,0}$ and we understand that $\phi_{-1}=0$ in the case $k=0$. The three unknown constants $S_{\pm 1}$, $\tilde{S}_{-1}$ are the Stokes constants for the Painlevé I equation.

As we will see, a consequence of these equations is that the asymptotics of the instanton solution u_k to Painlevé I is determined by the solution $u_{k'}$, with $k'=k\pm 1$, and by the trans-series solutions $u_{k|1}$, which do not belong to the instanton sequence.

4.2 A study of the $u_{n|1}$ trans-series

We will now study the trans-series solutions $u_{n|1}$. It turns out that there are *three* different cases: $n=0$, $n=1$ and $n\geq 2$. We will now study them in detail.

For $n=0$, $u_{0|1}$ satisfies the same linear ODE than u_1,

$$-\tfrac{1}{6}u''_{0|1} + 2u_0u_{0|1} = 0, \tag{4.18}$$

which indeed has two linearly independent solutions: one of them, corresponding to u_1, is exponentially decreasing along the direction $\arg z=0$, $|z|\to\infty$. The solution corresponding to $u_{0|1}$ is *exponentially increasing* along this direction, and it is given by

$$u_{0|1}(z) = z^{-1/8}\mathrm{e}^{+(8\sqrt{3}/5)z^{5/4}}\mu_1(z), \tag{4.19}$$

where $\mu_1(z)$ is a formal power series in $z^{-5/4}$:

$$\mu_1(z) = \sum_{n\geq 0} \mu_{n,1} z^{-5n/4} \tag{4.20}$$

and we normalize again

$$\mu_{0,1} = 1. \tag{4.21}$$

The coefficients $\mu_{n,1}$ satisfy the same recursion than $u_{n,1}$, (1.7), with the only difference that we have a minus sign on the right-hand side. One immediately finds that

$$\mu_{n,1} = (-1)^n u_{n,1}. \tag{4.22}$$

Let us now consider $u_{1|1}$, which satisfies the linear inhomogeneous ODE

$$-\tfrac{1}{6} u''_{1|1} + 2u_0 u_{1|1} + 2u_1 u_{0|1} = 0. \tag{4.23}$$

It is easy to see that $u_{1|1}(z)$ has the following structure:

$$u_{1|1}(z) = z^{-3/4} \sum_{n\geq 0} \mu_{n,2} z^{-5n/4}, \tag{4.24}$$

and in addition $\mu_{2n+1,2} = 0$. The even coefficients satisfy the recursion

$$\mu_{2n,2} = -\sum_{l=0}^{n-2} \mu_{2l,2} u_{n-l,0} - \sum_{l=0}^{2n} (-1)^l u_{l,1} u_{2n-l,1} + \frac{25}{192}(2n-1)^2 \mu_{2(n-1),2} \tag{4.25}$$

and we find, for the very first terms,

$$u_{1|1}(z) = -z^{-3/4}\left(1 + \frac{75}{512} z^{-5/2} + \frac{300713}{1572864} z^{-5} + \cdots\right). \tag{4.26}$$

For $k \geq 2$, the formal solutions $u_{k|1}$ develop a new feature: they contain logarithms. This is due to the resonant character of the Painlevé I equation. These solutions have the following form:

$$u_{k|1}(z) = \tfrac{5}{4} \log z\, g_{k-1}(z) + f_{k+1}(z), \quad k \geq 2, \tag{4.27}$$

where

$$f_k(z) = z^{1/2-5k/8} \mathrm{e}^{(2-k)Az^{5/4}} \mu_k(z), \qquad \mu_k(z) = \sum_{n\geq 0} \mu_{n,k} z^{-5n/4},$$
$$g_k(z) = z^{1/2-5k/8} \mathrm{e}^{-kAz^{5/4}} \nu_k(z), \qquad \nu_k(z) = \sum_{n\geq 0} \nu_{n,k} z^{-5n/4}. \tag{4.28}$$

The factor $\frac{5}{4}$ in (4.27) is introduced for convenience, since in the resurgent analysis it will be convenient to use the variable x in (4.5). The functions f_k and g_k appearing in (4.28) satisfy the coupled system of equations

$$-\frac{1}{6} g_k'' + 2u_0 g_k + 2\sum_{i=1}^{k-1} u_i g_{k-i} = 0, \qquad k \geq 1.$$
$$-\frac{1}{6} f_k'' + 2u_0 f_k + 2\sum_{i=1}^{k-1} u_i f_{k-i} + \frac{5}{24z^2} g_{k-2} - \frac{5}{12z} g_{k-2}' = 0, \qquad k \geq 3. \tag{4.29}$$

In the second equation, we set $f_1(z) = u_{0|1}(z)$ and $f_2(z) = u_{1|1}(z)$. It is easy to see, from the recursion relation obeyed by the coefficients $\nu_{n,k}$, that

$$\nu_k(z) = C k u_k(z), \qquad k \geq 1, \tag{4.30}$$

where C is a constant given by

$$C = \frac{16}{5A}. \tag{4.31}$$

The value of this constant can be fixed by looking at the equation for $u_{2|1}(z)$.

Finally, one can easily find recursion relations for the coefficients $\mu_{n,k}$ appearing in $f_k(z)$. The cases $k=3$ and $k \geq 4$ are slightly different. For $k=3$, one finds

$$\mu_{n,3} = \frac{8}{25A(n+1)} \left\{ -\frac{25}{64}(2n+1)^2 \mu_{n-1,3} + 12\sum_{l=0}^{n-2} u_{(n+1-l)/2,0} \mu_{l,3} \right.$$
$$\left. + 12\sum_{m=1}^{2}\sum_{l=0}^{n+1} u_{n+1-l,m} \mu_{l,3-m} + \frac{25}{16}(2n+1)\nu_{n,1} + \frac{25A}{8}\nu_{n+1,1} \right\}, \tag{4.32}$$

while for $k \geq 4$ we have

$$\mu_{n,k} = \frac{1}{12(k-1)(k-3)} \left\{ 12 \sum_{l=0}^{n-3} \mu_{l,k} u_{(n-l)/2,0} + 12 \sum_{m=1}^{k-1} \sum_{l=0}^{n} \mu_{l,m} u_{n-l,k-m} - \frac{25}{64}(2n+k-4)^2 \mu_{n-2,k} \right.$$
$$\left. - \frac{25}{16} Ak(k+2n-3)\mu_{n-1,k} + \frac{25}{16}(k+2n-4)\nu_{n-1,k-2} + \frac{25A}{8}(k-2)\nu_{n,k-2} \right\}. \tag{4.33}$$

One interesting aspect of the doubly indexed sequences $(u_{n,k})$, $(\mu_{n,k})$ is that we can also consider their dependence on k for fixed n, and it turns out that one can find closed formulae for these coefficients from the recursion. For example, $(u_{0,k})$ is given by

$$u_{0,k} = (12)^{1-k} k, \quad k \geq 1, \tag{4.34}$$

while $(u_{1,k})$ is given by

$$u_{1,k} = -\frac{12^{-k}}{16\sqrt{3}}(109k^2 - 120k + 24), \quad k \geq 2. \tag{4.35}$$

These formulae can also be obtained from the results in [5, Section 5.2]. Using their results one can see that, as a function of k, $12^k u_{n,k}$ is a polynomial in k of degree n. Finally, we give a general formula for $\mu_{0,k}$ when $k \geq 4$:

$$\mu_{0,k} = 12^{-k+1}(k-2)(141k-402), \quad k \geq 4. \tag{4.36}$$

Let us close this section with a problem dealing with the physical interpretation of the full trans-series (4.1).

Problem 4.1. As we mentioned in Section 1, the series $u_{n|0}(z)$ can be interpreted in terms of the double-scaling limit of instantons in the matrix model, and as an amplitude associated to a ZZ brane in Liouville gravity. What is the interpretation of the more general trans-series $u_{n|m}(z)$, with $m > 0$, in the context of matrix models and in the context of noncritical strings? Do they correspond to new nonperturbative sectors of these theories? □

5 Asymptotics

5.1 Asymptotics of the multi-instanton solutions

The asymptotics of the coefficients $u_{n,k}$ can be obtained from a well-known application of Cauchy's theorem in the Borel plane (see [7, 9, 17] for examples). To do this, we consider the formal power series ϕ_k appearing in the k- instanton solution, which we write in terms of the variable $x = z^{5/4}$ as

$$\phi_k(x) = x^{-\beta k} \sum_{n \geq 0} u_{n,k} x^{-n}. \tag{5.1}$$

Here, $\beta = \frac{1}{2}$. The Borel transform of $\phi_k(x)$ is

$$\hat{\phi}_k(p) = \sum_{n=0}^{\infty} \frac{u_{n,k}}{\Gamma(\beta k + n)} p^{n+\beta k-1}. \tag{5.2}$$

Therefore,

$$\frac{u_{n,k}}{\Gamma(\beta k + n)} = \oint_0 \frac{\mathrm{d}p}{2\pi \mathrm{i}} \frac{\hat{\phi}_k(p)}{p^{\beta k+n}}. \tag{5.3}$$

We now apply the standard deformation contour argument. The Borel transform $\hat{\phi}_k$ has branch cuts at $p = \pm A$, and we can deform the contour to encircle these (we also pick a vanishing contribution from a circle at infinity). But the contribution to the integral is given precisely by the discontinuity across the cut, that is, by the alien derivatives that we calculated in (4.16) and (4.17). We then obtain the asymptotic formula

$$\frac{u_{n,k}}{\Gamma(\beta k + n)} \sim_n \frac{S_1}{2\pi \mathrm{i}}(k+1) \int_A^{\infty} \mathrm{d}p \frac{\hat{\phi}_{k+1}(p)}{p^{\beta k+n}} - \frac{S_{-1}}{2\pi \mathrm{i}}(k-1) \int_{-\infty}^{-A} \mathrm{d}p \frac{\hat{\phi}_{k-1}(p)}{p^{\beta k+n}} - \frac{\tilde{S}_{-1}}{2\pi \mathrm{i}} \int_{-\infty}^{-A} \mathrm{d}p \frac{\hat{\phi}_{k|1}(p)}{p^{\beta k+n}}. \tag{5.4}$$

For $k \geq 2$ this formula involves the Borel transform of $\phi_{k|1}(x)$. To compute this, we have to use

$$\mathcal{B}(\log x\, x^{-\nu}) = -\frac{p^{\nu-1}}{\Gamma(\nu)} \log p + \frac{\psi(\nu)}{\Gamma(\nu)} p^{\nu-1}. \tag{5.5}$$

The calculation of the integrals appearing in (5.4) is very similar to the calculation in [17]. The only new ingredient is the logarithm appearing in the Borel transform (5.5), which leads to an integral of the form

$$\int_0^{\infty} \mathrm{d}\zeta \log \zeta \frac{\zeta^{r+\beta(k-1)-1}}{(1+\zeta)^{k\beta+n}} = \frac{\Gamma(r+\beta(k-1))\Gamma(n-r+\beta)}{\Gamma(n+\beta k)} (\psi(n-r+\beta) - \psi(r+(k-1)\beta)). \tag{5.6}$$

Note that the term involving $\psi(r+(k-1)\beta)$ will cancel against the contribution coming from the second term in (5.5). For n large we can use the asymptotic behavior

$$\psi(n-r+\beta) = \log n + \frac{\beta - r - 1/2}{n} + \mathcal{O}(1/n^2). \tag{5.7}$$

Before presenting the general result for the asymptotics derived from (5.4), let us analyze in some detail the case $k=0$, since this will fix one of the Stokes constants. For $k=0$ the only contributions to the asymptotics come from u_1 and $u_{0|1}$, and we obtain

$$\begin{aligned} u_{n/2,0} \sim{}& A^{-n+1/2}\frac{S_1}{2\pi \mathrm{i}}\Gamma\left(n-\frac{1}{2}\right)\left\{u_{0,1} + \sum_{l=1}^{\infty}\frac{u_{l,1}A^l}{\prod_{m=1}^{l}(n-1/2-m)}\right\} \\ &+ (-A)^{-n+1/2}\frac{\tilde{S}_{-1}}{2\pi \mathrm{i}}\Gamma\left(n-\frac{1}{2}\right)\left\{\mu_{0,1} + \sum_{l=1}^{\infty}\frac{\mu_{l,1}(-A)^l}{\prod_{m=1}^{l}(n-1/2-m)}\right\}. \end{aligned} \tag{5.8}$$

Since $u_{n/2,0}=0$ if n is not even, the right-hand side of this relation must vanish if n is odd. Using (4.22) we find that this is the case provided that

$$(-1)^{1/2}\tilde{S}_{-1} = S_1. \tag{5.9}$$

This is exactly as in the ODE studied in [6]. We then find the result [20] (see also [16])

$$u_{n,0} \sim A^{-2n+1/2}\frac{S_1}{\pi \mathrm{i}}\Gamma\left(2n-\frac{1}{2}\right)\left\{1 + \sum_{l=1}^{\infty}\frac{u_{l,1}A^l}{\prod_{m=1}^{l}(2n-1/2-m)}\right\}, \quad n\to\infty. \tag{5.10}$$

Note that S_1 has already been evaluated in (2.39) and it has the value

$$S_1 = -\mathrm{i}\frac{3^{1/4}}{2\pi^{1/2}}. \tag{5.11}$$

Let us now analyze $k=2$. The asympotics of $u_{n,1}$ involves u_0 and $u_{1|1}$. Using (5.9) we can write it as

$$u_{n,1} \sim A^{-n+1/2}\frac{S_1}{2\pi \mathrm{i}}\Gamma\left(n-\frac{1}{2}\right)\left\{2u_{0,2} + (-1)^n\mu_{0,2} + \sum_{l=1}^{\infty}\frac{(2u_{l,2}+(-1)^{n+l}\mu_{l,2})A^l}{\prod_{m=1}^{l}(n-1/2-m)}\right\}, \tag{5.12}$$

and it depends on the parity of n. Using the explicit results (1.18), it is easy to check that the leading asymptotic behavior is as stated by Joshi and Kitaev in [20, Proposition 16].

The formula (5.12) gives in addition all-orders expansion of $u_{n,1}$ as an asymptotic series in $1/n$.

We can now write a general formula for the asymptotics of $u_{n,k}$ when $k \geq 2$. Using again (5.9), and absorbing a factor $(-1)^{-1/2}$ in S_{-1}, we find

$$
\begin{aligned}
u_{n,k} \sim A^{-n+\beta} \frac{S_1}{2\pi \mathrm{i}} \Gamma(n-\beta) \left\{ (k+1) u_{0,k+1} + (-1)^n \mu_{0,k+1} + \sum_{l=1}^{\infty} \frac{((k+1) u_{l,k+1} + (-1)^{n+l} \mu_{l,k+1}) A^l}{\prod_{m=1}^{l} (n-\beta-m)} \right\} \\
+ (-1)^n (k-1) A^{-n-\beta} \frac{S_{-1}}{2\pi \mathrm{i}} \Gamma(n+\beta) \left\{ u_{0,k-1} + \sum_{l=1}^{\infty} \frac{u_{l,k-1} (-A)^l}{\prod_{m=1}^{l} (n+\beta-m)} \right\} \\
- (-1)^n A^{-n-\beta} \frac{S_1}{2\pi \mathrm{i}} \Gamma(n+\beta) (\log n - \log A) \left\{ \nu_{0,k-1} + \sum_{l=1}^{\infty} \frac{\nu_{l,k-1} (-A)^l}{\prod_{m=1}^{l} (n+\beta-m)} \right\} \\
- (-1)^n A^{-n-\beta} \frac{S_1}{2\pi \mathrm{i}} \Gamma(n+\beta) \\
\times \left\{ (\psi(n+\beta) - \log n) \nu_{0,k-1} + \sum_{l=1}^{\infty} \frac{\psi(n+\beta-l) - \log n}{\prod_{m=1}^{l} (n+\beta-m)} \nu_{l,k-1} (-A)^l \right\}
\end{aligned}
\tag{5.13}
$$

for $k \geq 2$, and we recall that $\beta = \frac{1}{2}$. As compared with the asymptotics for $k = 0, 1$, the asymptotics for $k \geq 2$ involves logarithmic terms. In fact, the dominant term in the asymptotics is precisely the $\log n$ term. In the last line, we use the asymptotics (5.7) for the ψ function. Finally, note from (4.30) that one has the relation (1.14) for the coefficients $\nu_{n,k}$.

5.2 Asymptotics of the multi-instanton solutions: numerical evidence

We will now perform numerical tests of the predicted asymptotic behavior (5.13) for the instanton series $u_{n,k}$. The standard technique to do that is the method of Richardson extrapolation. This method goes as follows. Let us assume that a sequence s_n has the asymptotics

$$
s_n \sim_n \sum_{k=0}^{\infty} \frac{a_k}{n^k}
\tag{5.14}
$$

for n large. Its *Nth Richardson transformation* $s_n^{(N)}$ can be defined recursively by

$$
\begin{aligned}
s_n^{(0)} &= s_n, \\
s_n^{(N)} &= s_{n+1}^{(N-1)} + \frac{n}{N} \left(s_{n+1}^{(N-1)} - s_n^{(N-1)} \right), \quad N \geq 1.
\end{aligned}
\tag{5.15}
$$

The effect of this transformation is to remove subleading tails in (5.14), and

$$s_n^{(N)} \sim a_0 + \mathcal{O}\left(\frac{1}{n^{N+1}}\right). \tag{5.16}$$

The values $s_n^{(N)}$ give numerical approximations to a_0, and these approximations become better as N and n increase. Once a numerical approximation to a_0 has been obtained, the value of a_1 can be estimated by considering the sequence $n(s_n - a_0)$, and so on.

The method of Richardson extrapolation can be applied *verbatim* to verify the asymptotics of $u_{n,1}$ written down in (5.12). Let us illustrate this with a nontrivial example. According to (5.12), the sequence

$$s_n = \left(2n - \frac{5}{2}\right)\left\{\left(2n - \frac{3}{2}\right)\left(\frac{u_{2n,1}\, A^{2n-1/2}}{S_1/(2\pi \mathrm{i})\,\Gamma(2n-1/2)} + \frac{2}{3}\right) - 2u_{1,2}\right\} \tag{5.17}$$

is of the form (5.14) and asymptotes, as $n \to \infty$, the value

$$(2u_{2,2} + \mu_{2,2})A^2 = -\frac{55}{96} \approx -0.5729166666666666\ldots. \tag{5.18}$$

In Figure 4, we show a plot of the sequence (5.17) and its first two Richardson transformations for n up to 200, which converges to the expected value (5.18). Taking

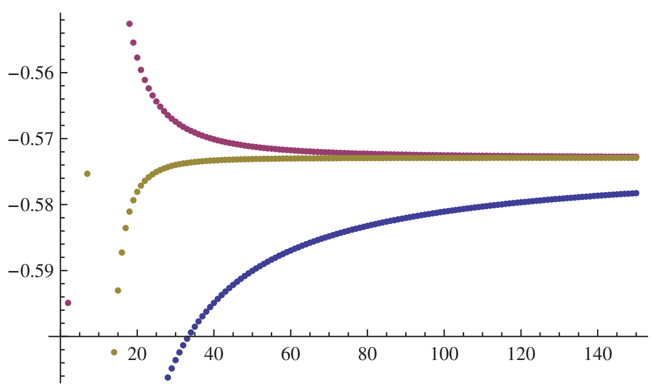

Fig. 4. A plot of the sequence (5.17) and its Richardson transformations. In this and subsequent plots, the horizontal axis represents the integer n, and the vertical axis represents the values of the sequence.

$n = 250$ and 10 Richardson transformations gives the numerical approximation

$$s^{(5)}_{200} = -0.572916666666666667\ldots. \tag{5.19}$$

Note that this test already verifies that the general trans-series solutions $u_{n|m}$ appearing in (4.1) are the relevant objects to understand the asymptotics, since $k = 1$ involves the trans-series $u_{1|1}$.

Let us now study the asymptotic behavior of the instanton sequences $u_{n,k}$ with $k \geq 2$. The main novelty here is the presence of logarithms in the asymptotics, and this does not fit *a priori* into the standard framework of Richardson transformations. However, one can transform the sequence and put it in a form which is amenable to an analysis with standard Richardson transformations, as pointed out in [28]. Let us assume that we have a sequence ℓ_m with the asymptotics

$$\ell_m \sim \log m s_m + t_m, \quad s_m = \sum_{k\geq 0} \frac{a_k}{m^k}, \quad t_m = \sum_{k\geq 0} \frac{b_k}{m^k}. \tag{5.20}$$

This type of asymptotc behavior appears in instanton corrections in Quantum Mechanics. The leading behavior of this sequence is determined by the coefficient a_0, and we would like to find a method to extract it numerically. To do this, we consider the sequence

$$\tilde{\ell}_m = m(\ell_{m+1} - \ell_m), \tag{5.21}$$

which has the asymptotics

$$\tilde{\ell}_m \sim \log m \tilde{s}_m + \tilde{t}_m, \quad \tilde{s}_m = \sum_{k\geq 1} \frac{\tilde{a}_k}{m^k}, \quad \tilde{t}_m = a_0 + \sum_{k\geq 1} \frac{\tilde{b}_k}{m^k}. \tag{5.22}$$

It is now easy to see that, if we apply the Richardson transformation (5.15) *twice* to this sequence, we remove both the tails in $1/n^p$ and the tails in $\log n/n^p$. This then allows a precise determination of the leading term a_0. Once this has been determined, we can extract the other coefficients in (5.20) by subtracting from the original sequence the parts of the asymptotics which are under control.

Let us now apply this idea to the sequence of instantons of Painlevé I. The first step is to consider the auxiliary sequence

$$a_{n,k} = \frac{A^{n+\beta} u_{n,k}}{\Gamma(n+\beta)} \tag{5.23}$$

whose leading asymptotics is

$$(-1)^n a_{n,k} \sim -\frac{S_1}{2\pi \mathrm{i}} \left\{ \nu_{0,k-1} - \frac{A\nu_{1,k-1}}{n} + \mathcal{O}(1/n^2) \right\} \log n + \frac{S_1}{2\pi \mathrm{i}} \log A\nu_{0,k-1} + \frac{S_{-1}}{2\pi \mathrm{i}}(k-1)u_{0,k-1}$$
$$+ A\left\{ \frac{S_1}{2\pi \mathrm{i}}((k+1)(-1)^n u_{0,k+1} + \mu_{0,k+1}) - \log A\nu_{1,k-1}) - \frac{S_{-1}}{2\pi \mathrm{i}}(k-1)u_{1,k-1} \right\} \frac{1}{n}$$
$$+ \mathcal{O}(1/n^2). \tag{5.24}$$

Again, it depends on the parity of n, that is, wether $n = 2m$ or $n = 2m + 1$. Let us denote

$$a^{(e)}_{m,k} = a_{2m,k}, \quad a^{(o)}_{m,k} = a_{2m+1,k}. \tag{5.25}$$

In both cases their asymptotics is of the form (5.20) and we can use the method of [28] to analyze the sequence numerically.

For simplicity, we will illustrate the asymptotics by focusing on the sequence with $n = 2m$ even, and we will test the four leading terms displayed in (5.24), that is, the (leading) term in $\log n$, and the terms in $\log n/n$, constant, and $1/n$. In terms of the general structure written down in (5.20), we will test the values of a_0, a_1 and b_0, b_1. These coefficients involve all the trans-series solutions appearing in the resurgent analysis.

We will first test the *leading term* of order $\log n$. Following [28], we first construct the sequence

$$\tilde{a}^{(e)}_{m,k} = s(a_{2m+2,k} - a_{2m,k}) \sim -\frac{S_1}{\pi \mathrm{i}\sqrt{3}}(k-1)^2(12)^{2-k} + \mathcal{O}\left(\frac{1}{m}, \log\frac{m}{m}\right), \tag{5.26}$$

where we have used the explicit value for $\nu_{0,k-1}$ derived from (1.14) and (4.34). To remove tails, we perform Richardson transformations in the sequence (each transform is performed twice to remove both types of tails, as explained above). In Figure 5, we plot it for $k = 2, 5$, together with the second and the fourth Richardson transformations, and for $m = 200$. The convergence towards

$$-\frac{S_1}{\pi \mathrm{i}\sqrt{3}} \approx 0.068228352037086\ldots \tag{5.27}$$

for $k = 2$, and towards

$$-\frac{S_1}{108\pi \mathrm{i}\sqrt{3}} \approx 0.00063174400034\ldots \tag{5.28}$$

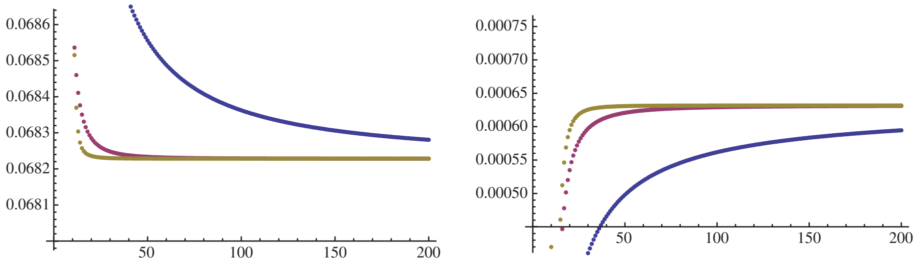

Fig. 5. A plot of the sequence (5.26) and its Richardson transformations, for $k=2$ (left) and $k=5$ (right).

for $k=5$, is manifest in the figures. More precisely, for $m=250$ and with $N=10$ Richardson transformations, we obtain the numerical approximations

$$\begin{aligned} \tilde{a}^{(e),(10)}_{250,2} &= 0.068228352037087\ldots, \\ \tilde{a}^{(e),(10)}_{250,5} &= 0.00063174400031\ldots. \end{aligned} \tag{5.29}$$

We next test numerically the coefficient of $\log s/s$. We consider the sequence

$$b_{m,k} = m\left(\tilde{a}^{(e)}_{m,k} + S_1\pi\mathrm{i}\sqrt{3}(k-1)^2(12)^{2-k}\right) \tag{5.30}$$

whose leading asymptotics is of the form

$$-\frac{4S_1}{5\pi\mathrm{i}}(k-1)u_{1,k-1}\log m + \cdots \tag{5.31}$$

so we can apply the above procedure. The sequence $\tilde{b}_{m,k}$ for $k=2,5$, and up to $m=200$, together with its second and fourth Richardson transformations, is displayed in Figure 6. In both cases we have convergence to the predicted values

$$k=2: -0.00426427200231\ldots, \qquad k=5: -0.000847589867\ldots. \tag{5.32}$$

For $m=250$ and with $N=10$ Richardson transformations, we obtain the numerical approximations

$$\begin{aligned} \tilde{b}^{(10)}_{250,2} &= -0.00426427200235\ldots, \\ \tilde{b}^{(10)}_{250,5} &= -0.000847589866\ldots. \end{aligned} \tag{5.33}$$

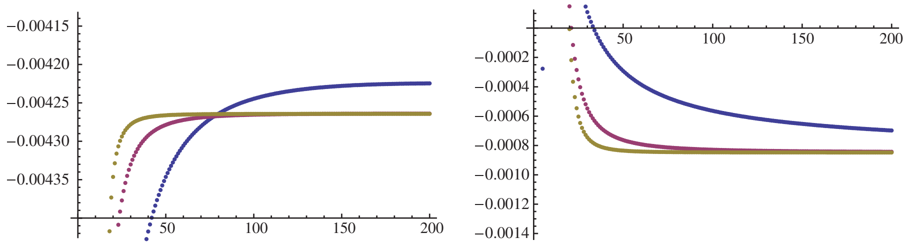

Fig. 6. A plot of the sequence $\tilde{b}_{m,k}$, obtained from (5.30), together with its Richardson transformations, for $k=2$ (left) and $k=5$ (right).

We can now study the constant term of the asymptotics, which also makes possible to obtain a numerical determination of the additional Stokes parameter S_{-1}. We consider the sequence

$$c_{m,k}=a_{2m,k}+\frac{S_1}{2\pi\mathrm{i}}\left\{\frac{2}{\sqrt{3}}(k-1)^2(12)^{2-k}-\frac{A\nu_{1,k-1}}{2m}\right\}\log(2m), \tag{5.34}$$

where we have subtracted the leading logs. According to the predictions of resurgence, as $m\to\infty$ this sequence asymptotes to

$$\frac{S'_{-1}}{2\pi\mathrm{i}}(k-1)^2(12)^{2-k}, \tag{5.35}$$

where

$$S'_{-1}=S_{-1}+\frac{2\log A}{\sqrt{3}}S_1. \tag{5.36}$$

The numerical analysis in the case of $k=2$ gives a numerical determination of the unknown Stokes constant S'_{-1} (hence, of S_{-1}), and we find, numerically,

$$\frac{S'_{-1}}{2\pi\mathrm{i}}\approx 0.31873285573864121\ldots \tag{5.37}$$

and consequently

$$\frac{S_{-1}}{2\pi\mathrm{i}}\approx 0.3882786818052856841\ldots. \tag{5.38}$$

Using this value, we can verify the asymptotic behavior (5.35) for the sequence (5.34) for higher values of k.

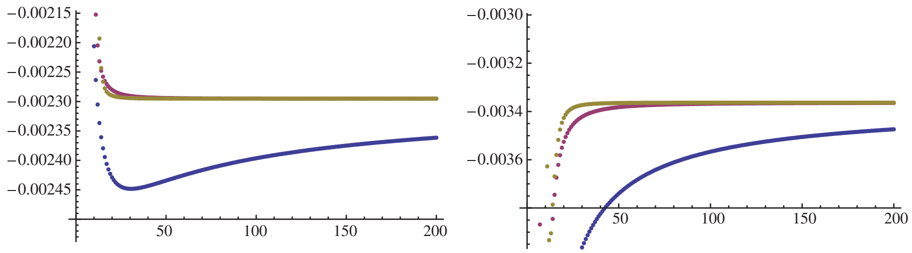

Fig. 7. A plot of the sequence (5.39) and its Richardson transformations, for $k=2$ (left) and $k=5$ (right).

Finally, we consider the coefficient of $1/m$. To do this, we construct the sequence

$$d_{m,k} = m^2(b_{m+1,k} - b_{m,k}), \tag{5.39}$$

which is asymptotic to

$$-\frac{A}{2}\left\{\frac{S_1}{2\pi \mathrm{i}}((k+1)u_{0,k+1} + \mu_{0,k+1}) - \frac{S'_{-1}}{2\pi \mathrm{i}}(k-1)u_{1,k-1}\right\}. \tag{5.40}$$

This involves the tran-series coefficients $\mu_{0,k+1}$, and we use the numerical determination of S'_{-1} obtained above. In Figure 7, we show the sequence (5.39) for $k=2$ and 5, up to $m=200$, together with its second and fourth Richardson transformations. They clearly match the predictions of resurgence,

$$k=2: -0.002295145874084\ldots, \qquad k=5: -0.0033633587118\ldots. \tag{5.41}$$

For $m=250$ and with $N=10$ Richardson transformations, we obtain the numerical approximations

$$\begin{aligned} d^{(10)}_{250,2} &= -0.002295145874083\ldots, \\ d^{(10)}_{250,5} &= -0.0033633587119\ldots. \end{aligned} \tag{5.42}$$

We believe that these numerical tests confirm in a very clear way the predictions from the resurgent analysis for $k \geq 2$.

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